ALGEBRA

A SECOND COURSE

The quality of the materials used in the manufacture of this book is governed by continued postwar shortages.

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A SECOND COURSE

BY

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ALGEBRA: A SECOND COURSE

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To

MRS. W. F. DURHAM

OF SHAWNEE, OKLAHOMA

the most inspiring teacher
I have ever known

PREFACE

This text is intended for students who have had one year of algebra. Because in many respects it represents a radical departure from the usual treatment of the subject, the reasons for the deviation must be presented in some detail.

The modern tendency in education is to give more attention to the viewpoint of the student and to recognize his right to enjoy the motivation that comes from an understanding of *purpose* and *significance* in advance of details. In no subject is this motivation needed more acutely than in algebra. Algebra is not multiplication or factoring—it is a way of doing things. It has purpose, logic, and usefulness. If these qualities are not demonstrated to the student, he is likely to find algebra uninteresting and difficult.

The algebraic method and its most significant advantages should be made clear to the student before he attempts to master the processes that are used in applying the method. When the processes themselves are given all the emphasis, when there is no general method to which the details can be related, the student repeatedly finds himself slaving away at operations whose significance is lost to him—even after he has learned to perform them. Expecting him to maintain enthusiasm under such conditions is like expecting a cake of ice to remain unmelted in midsummer sunshine.

The average student who begins his second course in algebra has forgotten much of what he learned during the first course; that this is true is evidenced by the amount of space devoted to a review of first-year algebra in all second-year textbooks. He has a vague notion that algebra may possibly be useful to some people, perhaps even to himself, but he has little idea of *how* it may be useful. He remembers "fiddling around" with x's and y's and equations, and he recalls with a shudder the so-called "written" or stated problems that he sometimes solved but rarely understood. Perhaps the most apt description of his impression of algebra is that it is like a kind of medicine that must be taken because the doctor so orders. In fairness, it must be admitted that many students do not have this attitude. Many teachers of first-year algebra are able to lead their students into an early appreciation of the

significance of the material, its completeness and logic—even while using a textbook in which details and processes are considered at length before the objectives connected with their use are delineated. Such teachers will be the first to welcome a presentation of intermediate algebra that is in line with their own methods.

Now consider the psychology that is most often employed in starting the student in his second course in algebra. First, he faces a comprehensive, systematic, step-by-step review of first-year algebra, relearning the same material, learning to add, subtract, multiply, and divide x's and y's. If it seemed dull before, it will seem more so now. Next comes the solving of equations and stated problems, and so on. The student does not even know where he leaves first-year algebra and takes up the second year. As for the purpose of algebra, he learns only that it is used to solve problems. He works some stated problems, but he thinks that expressions to be factored, equations to be solved, etc., are just as fundamental as stated problems. He is not shown why algebra is superior to other methods of solving problems, nor why it is indispensable to science and engineering. Moreover, if he finds algebra difficult when other subjects are not, he says he "doesn't have a mathematical mind," and lets it go at that, without knowing what constitutes a mathematical way of thinking and how he might improve himself in that respect.

This textbook is intended to remedy the difficulties just described by achieving these objectives:

1. To capture the interest of the student at the start by letting him examine his own mental skills and showing him how he can improve those which are essential to learning, particularly those essential in the learning of mathematics and science. This is the objective of Chap. 1, which exploits the universal interest in puzzles and riddles to illustrate, test, and improve the thought processes under consideration. It is the writer's observation that the material in Chap. 1 serves also to bolster the confidence of the student in his own mental abilities.

2. To demonstrate the algebraic method of analyzing practical problems so that the student understands what his objectives are *before* he begins the review of the fundamental operations of algebra. In Chap. 1, the student is made familiar with the trial-

and-error method of solving problems. He learns how to be systematic enough to make the best use of the trial-and-error method when it must be used. Then, in Chap. 2, he is shown how algebra eliminates the guesswork of the trial-and-error method by making it possible to proceed directly to the correct solution instead of "trying out" various answers to see if they "fit." Here is the answer to the fundamental problem involved in any teaching situation—how to motivate the student.

There are two essentials in the production of proper motivation: the arousing of interest (Chap. 1) and the demonstration of the purpose and the usefulness of the subject (Chap. 2). The student is prepared for a review of the fundamental processes of algebra when he sees that the solution of a problem by means of algebra involves two steps: (1) obtaining the proper equations and (2) solving the equations. *He first learns to obtain the equations*, finding that deriving equations from stated problems is not at all difficult (this point will be discussed later), then he learns that in order to solve equations it is necessary to utilize the fundamental algebraic processes that he studied in first-year algebra, but has now largely forgotten. He understands the superiority of the algebraic method, and he has a good idea of its practical value. Moreover, he understands that the problems met in practice are like stated problems; they are never set up as neat rows of equations to be solved, or as expressions to be factored. He is now completely motivated, and his momentum will carry him through Chap. 3, the review chapter, where he recognizes addition, multiplication, etc., as tools that are useful in solving equations; but he does not face the irksome necessity of thinking that he is learning such processes for themselves alone. Students must not be expected to slave away cheerfully on the fundamental algebraic processes without being given a complete picture of their relation to the rest of the subject.

In Chap. 4 the student takes up the solution of equations, completing his picture of the algebraic method of solving stated problems. From Chap. 4 on, this text differs from the usual one in only two significant respects:

1. Each new topic is introduced in terms of its purpose and its relation to the preceding material, before the processes involved in its use are examined in detail.

2. In order to achieve the first objective, it has been found necessary to devise new treatments for many topics. In each case, an attempt has been made to devise a presentation that fosters understanding from the beginning, rather than a memorizing process followed later by understanding.

There is a significant deviation from the usual presentation of methods for obtaining and solving equations. In this text, the material is presented in the following order:

1. Translation of stated problems into equations, by choosing a symbol for each quantity to be determined and restating the conditions of the problem in the form of equations.
2. Solution of simple equations in one unknown.
3. Solution of systems of equations in two (and three) unknowns.
4. Solution of stated problems in one, two, or three unknowns, by the method of (1) and (3).
5. Explanation that when one of the relations between the unknowns is very simple one unknown can be eliminated by expressing it in terms of the other at the start, thus saving several steps in the method of elimination by substitution.
6. Demonstration of situations in which the method of (5) is of greatest value.
7. Distinction between problems involving only one unknown and problems in which only one of two unknowns is to be determined.

It is significant that in the usual treatment the student is expected to solve stated problems in two unknowns by the method of (5), which constitutes *mental* elimination by substitution, before he has heard of elimination by substitution, or, for that matter, of equations in two unknowns. The general practice is to disguise problems in two unknowns by means of algebraic representation, postponing the consideration of pairs of equations until a good many stated problems have been solved. This practice is mainly responsible for the difficulty encountered in connection with stated problems. Before the student has solved a system of two equations, he is not prepared to understand a problem in two unknowns. At the same time, there is no point in assigning bona fide problems in one unknown; for such problems are mere exercises in arithmetic, requiring no algebra at all for their solution.

The *logical* way to analyze a problem in two unknowns is to assign a symbol to each unknown and to express each of the two necessary relationships between the unknowns as an equation. Stated problems cause no difficulty when approached in this fashion, and the usual aversion to them is avoided completely. The method of (5) should be presented as a special case, to be used whenever possible because of its convenience; but it should *follow* the general method so that it will not interfere with understanding of the fundamental relationship between problems and equations.

In the interests of continuity, other deviations from the usual order have been made. The study of imaginary numbers is delayed until the student is prepared to understand both their usefulness and their relation to systems of equations. Subjects that represent tools to be used in later studies, but which are not essential to the progressive development of the over-all picture, are placed in Chaps. 17 to 21, leaving the first sixteen chapters as an integrated unit.

Functional notation is deliberately avoided in connection with the graphical solution of equations in one unknown. In plotting $x^2 - 4x + 2$ as a function of x in order to solve the equation $x^2 - 4x + 2 = 0$, the student should not be handicapped by any symbol, such as $f(x)$ or y, which shifts his attention from the function $x^2 - 4x + 2$ itself. Likewise, the use of $f(x)$, etc., in the formal statement of principles or theorems, though an elegant shorthand, is a dangerous luxury that cannot be afforded by the student until he is well grounded in fundamentals. The use of functional notation to distinguish between variables and arbitrary constants in an equation is easily understood, however, and is presented (in Sec. 52) as a subject that may be included or omitted, at will.

The effect of Chaps. 1 and 2 upon student morale should not be underestimated. These chapters should be thoroughly digested before the study of Chap. 3 is begun. Attention should be given to the questions that appear at the ends of some of the chapters, particularly those near the beginning of the book. These questions are included because they cover points that must be made clear to the student before he goes on, if he is to be properly prepared for what he takes up later.

This text embodies principles utilized by the writer for a number of years at Oklahoma Baptist University. Each departure from the beaten path has been proved worth while in the classroom and found to pay generous dividends in student interest and understanding. Two objectives, to interest the student and to keep him aware of the entire plan of the algebraic method, are the keynotes of the treatment.

Principal acknowledgments are due Prof. Philippe Emmanuel LeCorbeiller, lecturer at Harvard University, whose many helpful suggestions and keen judgment have contributed immeasurably to the book; and Kenneth Eyer, instructor in physics at Oklahoma Baptist University, whose assistance in making up the answer list and ferreting out inconsistencies and errors in the manuscript has been invaluable. The suggestions and opinions contributed by others are greatly appreciated.

R. ORIN CORNETT.

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CONTENTS

CHAPTER 1

MENTAL GYMNASTICS

Involved in the learning of mathematics and science are a number of fundamental thought processes—basic mental skills which are more or less essential to all learning, but which form the very backbone of mathematics and science. In most fields one can get along fairly well without developing these specific abilities to more than a moderate degree. In mathematics and many branches of science and engineering, however, one's accomplishment is a direct function of the degree to which he develops them—together, of course, with the requisite amounts of time and effort devoted to their use. It is to be emphasized that all people of normal intelligence possess these abilities, at least in latent form, but that in a majority of cases they are not at all well developed. *The opportunity for conscious development of your basic mental skills is the challenge of this chapter, for in it you will learn what they are and how you can improve them.*

In the example that follows, and also in those presented later, it is of utmost importance that you perform each step exactly as directed. To investigate your own mind successfully, you must pay meticulous attention to every detail of the instructions. Throughout the book a double asterisk** will be used to mark the places where you must stop reading and do as you have just been instructed, or think about what you have just read. In each example, read all the instructions up to the double asterisk; *then stop reading.*

1. Visualization. Test your ability at *visualization* with this problem:

A certain cube is 3 in. on each side, thus having a volume of 27 cu. in. It is painted green and then sawed into 27 one-inch cubes. How many cubes have no sides painted, one side painted, two sides painted, etc.?

In attacking this problem, first *close your eyes* and make a mental picture of the 27 one-inch cubes stacked together in the form of the original 3-in. cube. *Be sure to close your eyes,* and concentrate upon your mental picture of the cube until it is so real you

can imagine yourself touching it, removing one or more of the 1-in. cubes, or looking at it from the front, back, side, top, or bottom. When your mental picture is so real that you can "see" the cubes in every detail, count the numbers of cubes with no sides painted, one side painted, two sides painted, etc. Do not use pencil and paper—keep your eyes closed. When you have finished, add your answers to make sure that the total is 27; then check with the answer at the bottom of page 4.**

If your answers to this example are correct and were obtained without a great deal of effort, you are extremely good at visualization, or the formation of mental pictures. The ability to project on the screen of one's mind a clear mental picture of an object, a scene, or the implications of a group of related ideas is extremely useful. Everyone can close his eyes and "see" a familiar room in his home, can enumerate and describe various objects therein, and can state the location of each. Likewise, everyone can remember the route to some familiar spot, "seeing" each turn and many of the buildings along the way. Such mental pictures are formed gradually, without effort; yet they can be formed almost immediately by the application of conscious effort. For example, look at the opposite side of the room, or look out of a window. Examine what you see for a few seconds; then close your eyes and examine your mental picture of it.** Notice as many objects or characteristics of the scene as are visible in the mental picture; then open your eyes and locate the missing portions.** Repeat this process until you have filled in the details to form a complete mental picture of the scene.**

Observation and study have shown that people who learn quickly and well, who make good grades without studying more than a reasonable amount, are invariably good at visualization. They remember things by "seeing" them in their minds. Perhaps you, when trying to remember something, can practically "see" the page where you read it. Unfortunately, most people are about 10 per cent efficient at visualization; *i.e.*, they do about 10 per cent as well at forming and remembering mental pictures as they could if they put their minds in shape and kept them that way. The surprising truth is that a few minutes of practice each day for a week or so will bring practically anyone up to par in this respect. For one who is extremely poor at visualization, two or

three times as much practice may be necessary before great improvement is observed, but the results are even more striking than in the average case.

2. Manipulation of Mental Images. These two problems will test your ability to *manipulate* mental pictures:

Picture the two numbers 567 and 785 in your mind. Concentrate on them until you can see them very clearly.** Now *close your eyes* and mentally add the two numbers. Keep your eyes closed during the addition, or at least do not look at the numbers—look at your mental pictures of them.**

Close your eyes and imagine an 8- by 12-in. sheet of white paper in front of your face in a vertical position. Mentally, lift the lower right corner and fold it over in such a way as to line up the bottom edge of the paper with the left edge.** Next, fold down the top 4 in. of the paper and cut along both folds.** You now have three pieces of paper. Arrange the three pieces of paper with their *shortest* sides in a straight line, and determine how long a line they form.**

The ability to manipulate and rearrange mental pictures enables one to avoid a great deal of the labor involved in, say, designing a house or a piece of radio equipment, painting a picture, or arranging the interior of a room. If the preliminary stages of the work can be done in the mind, much time, labor, and material can be saved. Too, there are many problems in mathematics, science, and everyday life that can be handled efficiently only if one is able to picture his efforts at a solution and to repeat and rearrange them until he can see his way to a successful conclusion.

Example 1. An artist is drawing a picture. As he draws, he visualizes the completed picture, changing this mental picture and rearranging it until it portrays the idea he wishes it to convey. By this method he does a minimum of repeated drawing, erasing, and redrawing.

Example 2. A radio engineer is designing a piece of apparatus that must be constructed to fit into a very small space in an army tank. After he has completed the electrical design, there remains the task of arranging the parts for the final assembly, which must be planned down to the last screw. The more capable he is at *mentally* arranging and rearranging the parts and changing the design to produce greater efficiency and compactness, the fewer times he will have to construct and reconstruct the apparatus, upon finding that something he has already mounted or planned is in the way of something else. If he can *mentally* bore holes

and install parts, rearranging them as needed, he can save much material and time.

3. Retention of Mental Pictures, Memorizing. This exercise is designed to test your *retention* of mental pictures:

Turn to the bottom of page 6 and examine the series of letters and numbers there for a few seconds, then *close your eyes* and picture them in your mind. Examine the series again to fill in the missing letters, until you can see all of them, from first to last.**

The easiest and most efficient way to memorize formulas, words in a foreign language, radio diagrams, poetry, or names and faces is to "see" them; *i.e.*, to form and retain mental pictures of them. During childhood and adolescence there is a period, known as the "golden memory age," during which one's mind is particularly adept at forming mental pictures and is equally good at retaining them. This is due in part to the fact that one's imagination is very active during those years when he spends a great deal of time daydreaming and pretending. When a child plays that he is flying an airplane, he "sees" the world from his airplane, he hears its engines, he lives through the experiences of flying as he conceives it. He experiences fantastic adventures, pictured in greatest detail. All this mental activity develops a great ability to picture things in the mind and to retain such impressions for long periods of time. As one grows older, the pressure of everyday affairs and the complete organization of his time into school, work, recreation, and sleep may keep him from giving free rein to his imagination often enough to maintain the same degree of skill at forming and retaining mental pictures. Thus, his powers of visualization and picture retention decline with time, unless preserved by something about his habits of thinking and working, or by the use of suitable mental exercises. This is responsible, at least in part, for the fact that many persons find themselves unable to memorize well after they reach college age or older. Since this

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Answers to the problem of the cubes:

1 cube with no sides painted
6 cubes with one side painted
12 cubes with two sides painted
8 cubes with three sides painted

27 cubes in all

ability is important in all branches of learning, it is of advantage to restore and refresh one's skill at forming, manipulating, and remembering mental pictures.

Now check up on yourself—can you remember the pattern of letters you learned a short time ago?** Can you remember the two numbers you added mentally in the preceding section?** Or can you remember the details of how you folded the piece of paper and cut it?**

Perhaps you do remember all these things, but perhaps also you will have forgotten them two or three days from now. If, however, you should refresh your mental picture of them tomorrow and the next day, you would be able to reproduce them 6 months from now. It is surprising how rapidly your ability to memorize will improve if you use this method--applying it, of course, to things you need to memorize, rather than to meaningless groups of letters.

In drawing a picture one looks repeatedly at the subject, then at the drawing, and back at the subject, in order to compare them in every detail. Likewise, in forming a *mental* picture, one must compare that mental picture in every detail with the original. Unfortunately, many people try to memorize material by reading or examining it over and over, *without* checking up on the mental record that is being formed. A mental picture can be useful only if it is accurate; hence one must exercise great care in forming it. Also, at least until one has had a great deal of practice, one finds it necessary to refresh mental pictures a few times to "make them stick."

4. Systematic Thinking. Use this problem to see if you think systematically:

You are to transport a fox, a chicken, and a basket of wheat across a river, taking only one of the three with you at a time. Can you do it without allowing the fox to eat the chicken or the chicken to eat the wheat?**

Now try a more difficult problem of the same type, as follows:

Three white men and three cannibals wish to cross a river. A motorboat that carries only two people is available. All three white men and one of the cannibals know how to operate the boat. Though the cannibals will give their complete cooperation in trying to get across the river, they will seize on any opportunity to attack the white men at an advan-

tage, *i.e.*, two cannibals to one white man, or three to two. Can you plan a program that will get them all across without affording the cannibals an opportunity for a "white-man stew?" Be careful—you are likely to think you have the solution long before you actually do; so do not check with the answer (which is explained below) until you can explain your solution to someone else. Don't give up easily—it may take you as little as 5 min. or as much as an hour. Use coins of one kind for the white men and coins of another kind for the cannibals; turn "heads up" for the ones who can operate the motorboat.**

No doubt you had quite a struggle with the preceding example, because it seems very difficult to most people. Its solution requires the application of *systematic thinking*, an ability that is extremely useful in any business or profession. Though there are a few situations in the problem where ingenuity and common sense are at a premium, there is little to guide one in solving it; so he must just plunge right in and see what happens. For this reason it is called a *trial-and-error* problem, one that is solved by continued trial until a solution is achieved. Many problems require a combination of trial and error with other methods.

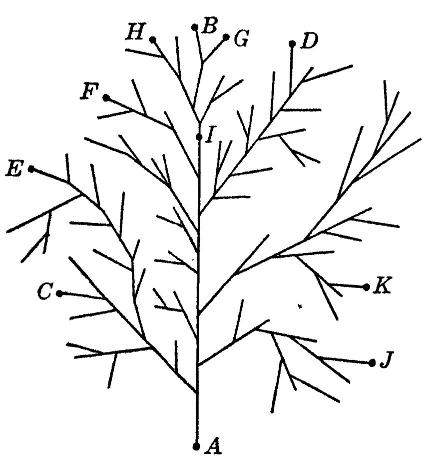


FIG. 1.—Pattern of a trial-and-error problem.

The average person goes over the same ground dozens of times in solving the problem of the cannibals and the white men; yet he would actually be forced into the correct solution *if he refused to start over* and if, when he was forced to abandon a blind alley, he would retreat *only to where he made his last choice.* Figure 1 represents the pattern of a trial-and-error problem. One starts at A, and the goal is at K. (Check the position of each letter as it is mentioned.) Starting out, one proceeds up the main branch, say, to B, and reaches a dead end. He starts again at A and veers off to the left, landing at C, another dead end. Starting once more at A, he wanders off to the right, reaching D. Reasoning: "I

guess it's on the other side," he tries again and lands at E, retreats to the main stem, then decides to investigate the branch leading to J. He may continue this procedure a long time before realizing how unsystematic it is, but finally he comes to his senses and starts a systematic search. Starting at A, he lands at B as before, but retreats only to the first intersection, where he investigates G, then H, and so on. Upon reaching I, he can be sure that the branch above him does not contain his goal. Next, he investigates the F branch, then the D branch, until he finds his goal at K. He may cover most of the "tree" in his search, but *he will not retrace his steps time after time.* This represents the systematic method of attacking a problem when there is little besides trial and error to use in solving it.

Let us apply this method to the preceding problem, making every mistake it is possible to make, *except* that of starting over aimlessly. Follow the solution with your coins. At the beginning it is seen that two cannibals (or one cannibal and one white man) must go across the river, and that a cannibal must be left on the other side, the remaining occupant returning with the boat. Next, the educated cannibal (who can operate the boat) must take the third cannibal across, leave him, and return with the boat. Any other course leads immediately to a meal for the cannibals. At this point, two cannibals are on the other shore, with the three white men and the educated cannibal still on this side. Now the only thing that can be done is to send two white men across. One of them must bring a cannibal back, or both must return; there is no other alternative. For both to return admits that their trip was of no avail; hence it is decided that one of them returns to the nearer shore, bringing a cannibal. After this return trip, there is no possible procedure except to send a white man and a cannibal back to the other shore—but this is obviously a retracing of steps. What is to be done? Long before this point is reached, one is tempted several times to give up and start over, and at this point it seems that he must. But if he refuses to do so until he has examined all the possibilities (and this is an excellent test of mental stamina), he realizes sooner or later that if he sends the *educated* cannibal and a white man to the other bank, then allowing the white man to return to this side with an ordinary cannibal, the two remaining white men can go

across and allow the educated cannibal to return for the other two cannibals, one at a time. Trace this solution, a step at a time, and you will find that at each point you are forced eventually in the right direction, if you refuse to retreat without examining every possibility.

5. Common Sense and Reasoning. This problem involves all the mental skills which have been described:

Suppose you are to select a certain number of eggs. Of these you must give half your eggs, plus $\frac{1}{2}$ egg more, to John. Next, you must give half the remaining eggs, plus $\frac{1}{2}$ egg more, to Paul. Finally, you must give half the remaining eggs, plus $\frac{1}{2}$ egg more, to Bill. You must now have one egg left, and all the eggs must be unbroken. How many eggs must you select at the start? Do not peek—work the problem before going on.**

Though this problem can be worked by trial and error, it is included here to illustrate the value of fully utilizing the information that is given in the problem. Instead of choosing different numbers and checking to see if they will work, start at the place where something is known—at the end. If you still have 1 egg, and if you have just given $\frac{1}{2}$ egg more than half your eggs to Bill, the single egg must be $\frac{1}{2}$ egg less than half the eggs you had; hence you must have had $1\frac{1}{2} \times 2 = 3$ eggs. Then, 3 eggs must be $\frac{1}{2}$ egg less than half the number you had before giving some to Paul; hence you must have had $3\frac{1}{2} \times 2 = 7$ eggs. Finally, 7 must be $\frac{1}{2}$ less than half the number you had initially; so you must have started with $7\frac{1}{2} \times 2 = 15$ eggs.

Choosing how to start a problem is often half the battle, for most problems that can be solved by simple trial and error can be worked much more easily if one examines the facts carefully and uses good judgment in deciding how to start, then proceeds systematically. A number of the problems at the end of this chapter provide opportunities for the use of good judgment and common sense. One develops these qualities by patient effort and care. Many problems must be worked by a combination of trial and error and careful deductions which indicate which way to go in making trial-and-error choices. In order to make full use of reasoning ability, one must proceed systematically in attempting a solution, as was shown in connection with the method

of trial and error. The use of trial and error in a *systematic* way, leading to consistent progress toward a correct answer, will henceforth be referred to as the method of *systematic trial.*

6. Concentration and Endurance. There is no better way to increase your powers of concentration and your mental stamina than to practice forming a complete mental picture of the interior of a room, the pattern on a rug, or a half page of poetry. In order to form a mental picture correct in great detail, you must achieve a degree of concentration that excludes everything else from your thoughts. To practice sustaining this degree of concentration for a full five or ten minutes will greatly increase your mental endurance.

Those who participate in athletic competitions realize the value of systematic exercises designed to develop and maintain a high degree of physical ability, coordination, and endurance, but all too few are aware of the corresponding need for exercises whose function is to develop and maintain a high degree of *mental* dexterity, efficiency, and stamina. The mind is capable of more rapid development and improvement than are any of the muscles of the body, and the rewards for such development are appropriately greater. It is unfortunate that in our schools more attention is given to saturating the mind with information than is devoted to training it to be skillful and efficient. Of course, the mind is developed a great deal in the process of acquiring knowledge—in school or out—but the degree of development attained is in most cases far inferior to that which can be brought about by a definite program of exercises designed to increase mental efficiency. No one denies, for example, that strenuous physical labor develops the body; but no one denies, either, that a much higher degree of well-rounded physical development results when work, even though physically strenuous, is supplemented by planned, systematic exercises and carefully chosen games.

There is nothing so wonderful as the human mind, the miracle of its memory, its understanding, its ability to reason and to deduce. It should be a source of much personal satisfaction to the individual student to come to close grips with his own mind, to become aware of those mental skills which he has developed, and to follow the training of his mind as an efficient, delicately controlled instrument that he can apply with confidence.

A great deal is made of those elusive mental characteristics

called *ingenuity*, *cleverness*, and *originality*, which enable one person to accomplish what to others may seem impossible. There is much evidence for believing that these characteristics are reflections of the basic mental skills that have been described in this chapter. For example, the person who solves the problem of the cannibals quickly and without hesitation might be called clever, but only because he has developed systematic thinking habits and has confidence in himself. The courage to attempt something entirely new and different is usually the result of a systematic attempt to succeed by usual methods. When these fail, it is concluded that a new method is required. A careful survey of all possibilities may then lead to some new method that will be called ingenious or original. Only the person whose thinking is systematic will have the necessary confidence in his conclusion that a new method is required in a given situation.

Visualization, mental agility, good memory, concentration, mental stamina, and the other mental skills that have been described all contribute to mental efficiency and power, generating the confidence that is necessary to originality and cleverness. Do not think of your mind as a fixed endowment—recognize it as a living, growing, developing instrument *that you can improve* from day to day. If you wish to do your best to develop those mental skills and abilities which are most essential in learning and in applying what is learned, you should practice the exercises at the end of this chapter regularly, for at least 10 or 20 min. each day for a week or two, or until you can tell that you are reaching your peak. The problems will help you cultivate systematic thinking habits and the ability to analyze and reason. This effort will help you in all your other studies, as well as in algebra.

EXERCISES AND PROBLEMS (6)

In the following exercises and problems, make sure that you do all mental work with your eyes closed.*

1. Mentally picture the numbers 478 and 356.** Close your eyes and add them.** Now, subtract them.** (If you find this exercise too difficult, start with two-digit numbers.)

* Problems 5, 6, 7, 8, 14, 15, 16, and 17 are reproduced (with slight changes) by permission of the author, from "A Book of Modern Puzzles," by Gerald L. Kaufman, published by Simon and Schuster. This book is a veritable treasure house of novel and fascinating puzzles.

2. Picture the numbers 257 and 24.** Close your eyes and multiply them.**

3. Picture the numbers 87, 42, and 63, in a vertical column.** With your eyes closed, add them.**

4. Use the mental addition of columns of numbers, the multiplication and division of numbers, etc., as a regular exercise. It will do wonders for your visualization, memory, and other mental powers. Strive to manipulate larger and larger numbers as your ability increases. This one exercise can be a most significant factor in your mental development.

5. Mentally, turn your left glove inside out and put it on your right hand. Is the palm of the glove next to the palm or the back of your right hand?

6. You are inside a barbershop. The word "barbershop" is painted on the window so as to be read *from the street.* Does the lettering appear correct or reversed to you if you see it in a mirror that reflects it from another mirror that reflects it from a third mirror?

7. Visualize a sheet of paper. Mentally, fold this paper to make a crease running from the upper right corner to the lower left corner. Now hold the paper *upside down* and *backside foremost,* and determine whether the crease runs (1) from upper right to lower left or (2) from upper left to lower right.

8. You are standing in front of a mirror with a piece of thin glass, 8 by 8 in., in one hand. On the glass you have painted the letter E. You are holding the glass so that, to you, it appears that your image in the mirror is holding the glass in his left hand and that the letter E is in correct position for him to read. In which hand are you actually holding the glass; and how does the letter E look to you when you look at it in your own hand; *i.e.,* not in the mirror?

9. Two books are side by side in front of you on a shelf. The left-hand book, which has 340 pages, is upside down. The right-hand book has 421 pages. If you add the number of the page at the extreme right of the right-hand book to the number at the extreme left of the left-hand book, what is the result?

10. Form a mental picture of an equilateral triangle whose sides are 2 in. long.** Now picture a circle inscribed in the triangle; *i.e.,* just touching the triangle at the mid-point of each of the sides.** Notice that when you concentrate on the circle its image becomes clearer and the triangle fades a bit, and vice versa.** Next, picture a second equilateral triangle inscribed in the circle, just touching it at three points, and continue this process of putting triangles in each successively smaller circle, and vice versa, until they become very small.** Try not to lose any of the images of the first triangles and circles as you add more.

11. Imagine a sheet of paper 11 in. high and $8\frac{1}{2}$ in. wide. From

the upper left corner, a point is reached by moving $3\frac{1}{2}$ in. down, and $2\frac{1}{2}$ in. to the right. Close your eyes and locate this point, mentally. Another point is reached from the lower right corner by moving $4\frac{1}{2}$ in. to the left, and 4 in. up. Mentally, determine how far up or down, and how far to left or right, the first point is from the second. This is a fairly difficult problem if you do it with your eyes closed. Do not look at a sheet of paper and trace your images on it—that would spoil the problem.

12. The section of track MN in Fig. 2 is a section of the main line of a railroad. There are boxcars A and B on the Y, and an engine E is on the main line. The spur S is very short, just long enough to hold one boxcar, but not the engine or two boxcars. The problem is to use the engine to interchange the two boxcars, then get the engine back on the main line. No "flying switches" or tow lines are allowed. You may keep your eyes open on this one—in fact, you should use coins or other objects to represent the cars and the engine.

NOTE: Problem 12 and those following are quite difficult, but very interesting. Do not be discouraged if you fail to solve them.

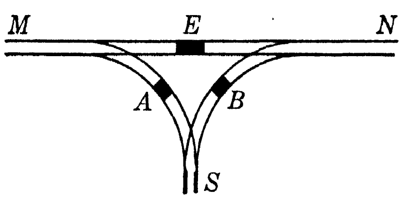
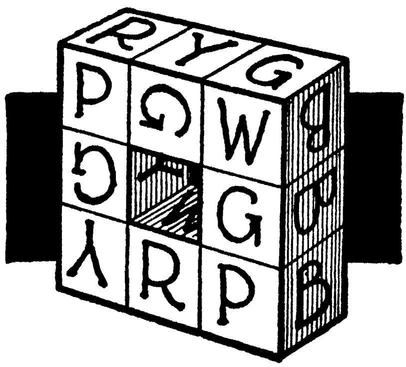


FIG. 2.—The problem of the Y.

13. Two dimes are placed side by side, flat on a table, with their edges touching. If one is kept stationary and the other is rolled around it (without slipping) once, how many times does the moving dime rotate on its axis? Solve the problem *mentally;* then check your answer, using dimes with well-milled edges to prevent slipping. HINT: Be sure to visualize the figures on the dimes (say, heads), in order to facilitate your visualization of the rotation. This problem is very difficult to solve mentally.

14. THE COLORED BLOCKS

The letters are supposed to represent six colored sides:
WHITE, BLACK, GREEN, RED, YELLOW, PURPLE.



The blocks are identically colored and lettered, so that what is true for one is true for all. Roll them around in your mind and see if you can answer these questions:

1. What letter is on the back of the upper right corner block?
2. What letter is on the top of the lower right corner block?
3. What letter is on the bottom of the lower center block?
4. What letter is on the back of this lower center block?

QUESTIONS FOR ALL BLOCKS:

5. What letter is on the side opposite Y?
6. What letter is on the side opposite G?
7. What letter is on the side opposite W?
8. What four edges touch the side P?

15. SIX BATHING BEAUTIES

The six bathing beauties stood in a line facing the judges' stand. The prize was given to the only girl whose name began with the same letter as her state, but of course the judges did not realize this until the contest was over. From the eight statements below, you should be able to associate each girl's name with her state and her position in the line. This done, you can easily pick out the prize winner.

MISS OHIO wasn't on speaking terms with DOROTHY.
OLGA was engaged to MISS DELAWARE'S brother.
MARY and MISS MARYLAND were at opposite ends.
MISS KANSAS was between OLGA and MISS MAINE.
DOROTHY was at the judges' extreme right, next to MISS MAINE.
Neither MAUDE nor VERA represented OHIO.
MISS VERMONT was between KATIE and MISS DELAWARE.
VERA was not next to the girl at the judges' extreme left.

16. STEPPE BY STEPPE

A LITTLE ADVENTURE IN SOVIET GEOGRAPHY

1. It's as far from OMSK to UMSK as from AMSK to MONSK.
2. It's as far from MINSK to MUNSK as from MANSK to MONSK.
3. MONSK is on a straight road north from MINSK to UMSK.
4. MONSK is on a straight road east from OMSK to MUNSK.
5. AMSK is 8 miles north of OMSK and 8 miles west of UMSK.
6. MANSK is 6 miles south of MUNSK and 6 miles east of MINSK.

HOW FAR IS IT FROM OMSK TO MINSK?................
HOW FAR IS IT FROM UMSK TO MUNSK?..............

To draw a diagram will help you a lot in solving this problem, particularly if you consider the top of the page as NORTH.

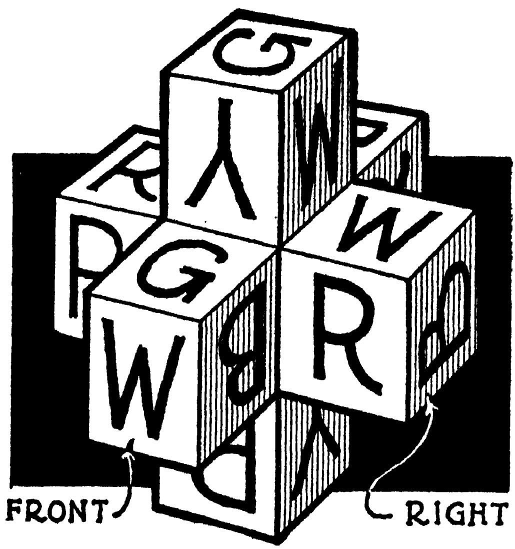
17. THE INSIDE BLOCK (VERY DIFFICULT)

The letters are supposed to represent six colored sides:

WHITE, BLACK, GREEN, RED, YELLOW, PURPLE.

All blocks are identically lettered, so that what is true for one is true for all. There are seven in the sketch, six being partly shown, and the seventh being

THE INSIDE BLOCK



This block has been placed in position so that NOT ONE OF ITS LETTERS TOUCHES THE SAME LETTER ON ANOTHER BLOCK.

Determine the letter on each of the faces of THE INSIDE BLOCK: top, bottom, front, back, left, and right.

Note: There are seven possible solutions for this problem.

CHAPTER 2

THE ALGEBRAIC METHOD

You are to buy 45 stamps, some of which are 3-cent, the rest 2-cent stamps, for a total cost of \$1.02. How many of each should you buy? Solve this problem by systematic trial.**

In solving this problem, you may have started by guessing that you should buy about 20 three-cent stamps and 25 (*i.e.*, 45—20) two-cent stamps, worth

$$(20 \times 3) + (25 \times 2) = 110 \text{ cents}.$$

Since this exceeds \$1.02, you have selected too many three-cent stamps; hence try 15 three-cent and 30 two-cent stamps, worth

$$(15 \times 3) + (30 \times 2) = 105 \text{ cents}.$$

Continuing in the same way, 10 three-cent and 35 two-cent stamps are worth

$$(10 \times 3) + (35 \times 2) = 100 \text{ cents}.$$

This sum is too small; so now try 11 three-cent and 34 two-cent stamps worth

$$(11 \times 3) + (34 \times 2) = 101 \text{ cents}.$$

Finally, 12 three-cent and 33 two-cent stamps are worth

$$(12 \times 3) + (33 \times 2) = 102 \text{ cents},$$

as desired. Note that this method of solution requires a combination of common sense with trial and error.

The *analytic* or *algebraic* method of solving problems is designed to replace systematic trial, wherever possible, by a procedure that leads directly to the correct answer or set of answers. Even simple problems like the foregoing are easier to solve by this method, and countless problems that could not be solved in months by trial require only a few minutes when attacked by the algebraic method. Algebra is one of the most important tools of the mathematician, the scientist, and the engineer. As such it is worth learning, and worth learning well; yet, as an agent in the development of habits of clear perception, systematic thinking, and

straightforward analysis of situations or problems, it is even more important; it is the open road to mental efficiency.

7. Algebraic Translation. In order to solve a stated problem such as the foregoing by the algebraic method, it is necessary first to translate the ideas of the problem into mathematical sentences, or equations. This process, called *algebraic translation*, consists of two simple steps: (1) the choosing of letters or symbols to represent each of the quantities (numbers) to be determined in the problem; and (2) the restatement of the ideas of the problem in the form of mathematical sentences (equations), in which the letters chosen are used instead of words. Since practically all mathematical sentences consist of the simple statement that something is equal to something else, and since the problem always contains the information needed to state what equals what, algebraic translation by means of the two steps listed above is not at all difficult.

In the preceding problem there are two numbers to be determined: the number of 3-cent stamps and the number of 2-cent stamps. These could be called x and y, or anything you like; but to make them easy to remember, let us call them N_3 (N for number, 3 for 3-cent stamps) and N_2. Now consider the first part of the stated problem: "You are to buy 45 stamps, some of which are 3-cent, the rest 2-cent stamps. . . ." This part of the sentence tells us that there are 45 stamps in all. We might write, then: "The number of 3-cent stamps plus the number of 2-cent stamps equals 45," or we might just as well write

$$N_3 + N_2 = 45.$$

Next, consider the part of the sentence, "for a total cost of \$1.02." This implies, "The value of the 3-cent stamps plus the value of the 2-cent stamps is \$1.02, or 102 cents." This statement can be written in mathematical symbols as $3N_3 + 2N_2 = 102$, since the value of N_3 3-cent stamps is $3N_3$ and the value of N_2 2-cent stamps is $2N_2$. Now we have two equations:

$$N_3 + N_2 = 45 \qquad \text{Equation (1)}$$
$$3N_3 + 2N_2 = 102. \qquad \text{Equation (2)}$$

Perhaps you remember how to solve to get N_3 and N_2 from these equations, but more likely you do not, since you probably took first-year algebra a long time ago. At the moment, we shall not

be concerned with solving these equations. Anyone can memorize the steps necessary to solve them—the point is that they can be solved very easily and that they yield the answers $N_3 = 12$, $N_2 = 33$. Checking these answers, 12 three-cent stamps are worth 36 cents, and 33 two-cent stamps are worth 66 cents. Then $36 + 66 = 102$, which is correct, and $12 + 33 = 45$, also correct. Note that there is nothing "hit or miss" about using these equations—*they give only one set of answers, the correct ones.*

It may surprise you to be told that if all the explanations and comments are omitted, it actually takes much less time to solve the stamp problem by algebra than by any other method. The most important thing, however, is that the algebraic method makes it possible to solve conveniently many problems that could not be solved at all by other methods. At the same time, it greatly reduces the labor involved in solving most problems that could be handled by methods involving trial and error.

Algebraic translation, as has been shown, transforms a practical problem into one or more equations that can be solved by routine steps (multiplying, adding, factoring, substitution, etc.), *i.e.*, by "turning the crank." The routine processes involved in "turning the crank" are the "tools" of algebra, just as the hammer and the saw are the tools of carpentry. Anyone can learn how to use a hammer and a saw, but that does not mean he will be able to build a beautiful piece of furniture. So it is that, although the tools of algebra are necessary, the real test is to make use of them. Algebraic translation makes it possible to apply the tools of algebra to important problems in all branches of engineering, science, and mathematics, as well as in countless other fields of endeavor.

Suppose that Harry and Dick divided a certain number of dollar bills. If Harry then gave one to Dick, they would have the same number. If, instead, Dick gave one to Harry, Harry would have twice as many as Dick. How many did each receive in the beginning? First, solve the problem by systematic trial.**

After you have first solved this example by systematic trial, see if you can go through the steps needed to translate the problem into one or more equations. Let H be the number of dollar bills Harry received in the beginning, and let D be the number Dick received. Do not peek! First, write the two equations by yourself; then follow the solution given below.**

In translating the problem into equations, you would first translate the sentence, "If Harry then gave one to Dick, they would have the same number." Now, if Harry gave one to Dick, he would have left one less, or $H - 1$, and Dick would have one more, or $D + 1$. But they have the same number after the transfer, so that $H - 1 = D + 1$ is the equivalent of the first sentence. Next, consider the second sentence, "If, instead, Dick gave one to Harry, Harry would have twice as many as Dick." If Dick gave one to Harry, Harry would have $H + 1$ and Dick would have $D - 1$. But Harry would have twice as many as Dick; hence $H + 1 = 2(D - 1)$. This equation is equivalent to the second sentence. Putting the 2 inside the parentheses gives $H + 1 = 2D - 2$. We now have the two equations:

$$H - 1 = D + 1 \qquad \text{Equation (1)}$$
$$H + 1 = 2D - 2 \qquad \text{Equation (2)}$$

These equations will now be solved, just for the sake of completeness, although the solving of equations is not to be studied until later. Subtract each side of equation (1) from the corresponding side of equation (2). This gives $0 + 2 = D - 3$. Now add 3 to each side, obtaining $5 = D$. This shows that Dick started with 5 dollars. Now let us rewrite equation (1), replacing D by its numerical value, 5.

$$H - 1 = D + 1 \qquad \text{Equation (1)}$$

Replacing D by 5, $\quad H - 1 = 5 + 1$

Adding 1 to each side, $\quad H = 7$

Thus Harry had 7 dollars in the beginning. Check the answers 5 and 7 by trying them out in the original problem.**

The importance of the algebraic method cannot be emphasized too strongly. For example, suppose that an engineer is designing a large suspension bridge, which is to cost half a million dollars. During his planning of the bridge, it becomes necessary for him to determine the length and size of a girder that is to fit into an important place. This girder will be swung into position during the construction of the bridge, and if it does not fit the place planned for it, the construction of the bridge will be delayed while another girder is being ordered, made, and delivered. More important, if the girder is not strong enough, or if it is too heavy, perhaps the entire structure will collapse. The engineer cannot afford to *guess* at the specifications of the girder, or to test his estimates by trial and error, because an error might be disastrous. He must make use of the algebraic method, along with the principles of structural engineering, to determine the proper length, weight, shape, and strength of the girder, *in advance. He must use equations to determine the correct specifications, eliminating guesswork.*

The person who expects algebra to be fully useful to him in science, engineering, or mathematics, or who wishes to incorporate the methods of algebra into his thinking habits, must achieve three objectives: (1) he must cultivate those mental skills described in Chap. 1; (2) he must learn to translate practical problems into equations; and (3) he must learn how to solve equations and perform the other routine operations of algebra.

EXERCISES (7)

Translate the following problems into equations, but do not solve the equations. Your equations need not contain the same letters given in the answers; choose any letters you like to represent the undetermined quantities. Remember that algebraic translation consists of two steps: (1) the choosing of letters to represent each of the quantities to be determined in the problem; and (2) restatement of the ideas of the problem in the form of mathematical sentences (equations), in which the letters chosen are used instead of words.

Write the equations, but *do not solve them.*

1. Find two numbers whose sum is 60 and whose difference is 14.

2. The sum of two numbers is 24, and one of the numbers is twice the other. What are the numbers?

3. A man has $3.95 in nickels and dimes. How many of each has he if there are 55 coins in all?

4. The gate receipts at a football game were $5,040 for 4,800 paid admissions. If reserved-seat tickets were $1.50 each and general-admission tickets were $.90, how many tickets of each kind were sold?

5. An airplane whose air speed is 225 m.p.h. requires 5 hr. to fly 1,000 miles against a strong head wind. What is the speed of the wind? HINT: The distance traveled equals the net speed multiplied by the time.

6. Two men have a total of $340. One of the men has $82 more than the other. How much does each have?

7. An automobile traveled 1,500 miles in two days. It traveled 135 miles farther on the second day than it did on the first. How far did it travel each day?

8. A grocer mixes two grades of coffee that he has been selling for 60 cents per pound and 40 cents per pound, respectively. How much of each must he take to make a mixture of 25 lb. to sell at 44 cents per pound? HINT: One of the equations should express the total value of the 25 lb. of coffee.

9. A man invested $10,000, part at 2% interest and part at 3%.

The investments yielded \$275 in a year. How much was invested at each rate?

10. A man made two investments totaling \$12,000. He profited 10% on the first but lost 5% on the second, so that his net profit was \$100. What was the amount of each investment?

11. A man invests \$2,000 more at 4% than he does at 3%. The total interest on the two investments amounts to \$351 per year. How much is invested at each rate?

12. Two cars were 200 miles apart after traveling in opposite directions for 4 hr., starting at the same point. If one traveled 5 m.p.h. slower than the other, find the speed of each.

13. Two hours after an airplane left an airport another plane started in pursuit of it. Six hours were required for the second plane to catch the first. If the second plane traveled 45 m.p.h. faster than the first, what were their speeds?

14. A man started to a town $52\frac{1}{2}$ miles away, walking 3 m.p.h. After walking part way, he was picked up by a motorist who took him to his destination at 30 m.p.h. If the entire trip required 4 hr., how far did he walk and how far did he ride? HINT: One of your equations must be a "time" equation.

15. One automobile went 20 miles farther, traveling 50 m.p.h., than a second one that traveled 3 hr. longer at 40 m.p.h. How many hours did each travel?

16. A man has 50 cents more than enough to buy 8 Meadowlark golf balls. He lacks 70 cents of having enough to buy 10 Longflite golf balls, which are 5 cents higher in price than the Meadowlark balls. Determine the prices of the two kinds of balls.

8. Practical Problems. The stamp problem analyzed in the preceding section is not what one would call a *practical* problem. It is often necessary to employ such made-up problems to illustrate the *methods* used in handling practical problems. For example, the methods used in solving many problems in radio and electrical engineering will be presented in the following chapters. In order to use actual problems in radio and electricity to illustrate these methods, however, it would be necessary to include several chapters explaining the principles and terms of radio and electricity, chapters that would consume extra time and be of use to only a small percentage of algebra students.

Where practical problems can be used as examples without making necessary a great deal of extra explanation, they will be

used. In all other cases, the problems used will *illustrate* practical methods, even though they themselves may seem as trivial in some cases as computing the number of whiskers in Grandfather's beard. The advantage of using such trivial problems is that they involve familiar situations that the student can analyze for himself and in so doing master the techniques used in solving the most practical of problems. *The importance and usefulness of the algebraic method make practical every exercise and problem that is assigned.*

It is a surprising fact that the student who has a good understanding of algebra at the level of this text is equipped mathematically to handle the majority of practical problems encountered in all branches of science and engineering. This does not mean that higher mathematics is not useful—quite the contrary—but only that elementary mathematics is extremely useful.

9. Formulas. In science and engineering there are practical situations which occur repeatedly, so that certain problems must be solved again and again, with new numbers each time.

Consider another stamp problem in which one is to buy 60 stamps, some of which are 3-cent and some 2-cent, for a total of \$1.34. Except for the number of stamps and their total value, this problem is identical with that at the beginning of the chapter. One could solve it by systematic trial or by translating it into the equations

$$N_3 + N_2 = 60 \qquad \text{and} \qquad 3N_3 + 2N_2 = 134$$

In fact, one could make a whole series of stamp problems: to buy 70 stamps worth \$1.82, 43 stamps worth \$1.14, 28 stamps worth 62 cents, and so forth, with N_3 and N_2 to be determined in each case. It would be foolish to set up two equations for each of these problems. Instead, one could write the two equations

$$N_3 + N_2 = T$$

where T represents the total number of stamps, and

$$3N_3 + 2N_2 = C$$

where C represents the total cost of the stamps.

These equations are exactly like the ones obtained in the first stamp problem, except that the letters T and C are used instead of the numbers 45 and 102. If one solves these equations for N_3 and N_2 (do not solve

them now), he obtains, not numerical answers, but answers expressed in terms of C and T, as follows:

$$N_3 = C - 2T \qquad \text{and} \qquad N_2 = 3T - C$$

These equations are *formulas* for N_3 and N_2.

A formula is an equation in which (1) there is only one quantity to be determined, and (2) the quantities on which it depends are represented by letters. By putting in the proper values for these letters, one can solve any problem of the type for which the formula is designed.

Numerical answers for the stamp problems can be determined from the formulas for N_3 and N_2 by simply putting in the values of C and T for the particular problem. For example, let us solve the first stamp problem (to buy 45 stamps worth \$1.02). Observe that in this problem the number of stamps (T) is 45, and that the total cost (C) is 102 cents. Then, using the formulas,

$N_3 = C - 2T$	$N_2 = 3T - C$
becomes	becomes
$N_3 = 102 - (2 \times 45)$	$N_2 = (3 \times 45) - 102$
$N_3 = 102 - 90$	$N_2 = 135 - 102$
$N_3 = 12$ three-cent stamps	$N_2 = 33$ two-cent stamps

Or consider the problem of buying 60 stamps worth \$1.34. Here $T = 60$ and $C = 134$, so that

$N_3 = C - 2T$	$N_2 = 3T - C$
becomes	becomes
$N_3 = 134 - (2 \times 60)$	$N_2 = (3 \times 60) - 134$
$N_3 = 14$ three-cent stamps	$N_2 = 46$ two-cent stamps

Since a formula expresses the desired quantity in terms of the quantities on which it depends, it is a rule of computation. Examples of useful formulas are listed below.

1. The area of a rectangle equals the product of its length and its width: $A = lw$.

2. The area of a triangle equals one-half the product of its base and its altitude: $A = \frac{1}{2}bh$ (b = base, h = altitude).

3. The circumference of a circle equals π times the diameter: $C = \pi d$. NOTE: $\pi = \frac{22}{7}$ for practical purposes.

4. The area of a circle equals π times the square of the radius: $A = \pi r^2$.

5. The distance traveled by an object moving at constant speed is the product of its speed of travel and the time: $D = ST$.

6. The amount of simple interest I on money invested equals the product of the principal P (the money invested), the rate R of yearly interest, and the time T in years: $I = PRT$.

7. The vertical distance (in feet) traveled by an object that is dropped is approximately equal to sixteen times the square of the time (in seconds) that it falls, if the distance of fall is not more than a few hundred feet: $D = 16T^2$.

Example 1. Find the area of a rectangle 12 ft. long and 8 ft. wide.

Solution: $A = lw = 12 \times 8 = 96$ sq. ft.

Note that a formula is used by substituting arithmetic numbers for the letters in the formula.

Example 2. Find the distance traveled by an airplane in 3.4 hr., if its speed is 225 m.p.h.,

Solution: $D = ST = 225 \times 3.4 = 765$ miles

PROBLEMS AND EXERCISES (9)

1. Determine the area of a circle whose radius is 3 ft.

2. Determine the circumference of the circle of Prob. 1.

3. The fall of a stone from the top of a tower is timed with a stop watch. The stone falls to the ground in 3 sec. How high is the tower?

4. The sum of \$3,000 is invested for 6 years at $2\frac{1}{2}\%$ per annum. How much interest accrues?

5. Determine the area of a triangle obtained by cutting a rectangle 6 by 8 ft. into two equal pieces.

6. The surface area of a sphere of radius r is given by the formula $S = 4\pi r^2$. Determine how much leather must be used to make a basketball 1 ft. in diameter if none of the leather is wasted in cutting.

7. What is the simple interest on \$3,400 invested at 3% per annum for $4\frac{1}{2}$ years?

Write the equations from which one could obtain formulas for the following. *Do not solve the equations.*

8. Find two numbers N_1 and N_2 whose sum is S and whose difference is D.

9. Find two numbers N_1 and N_2 whose sum is S, if one number is twice the other.

10. Find two numbers N_1 and N_2 whose difference is D, if one number is three times the other.

11. Find the amounts A_1 and A_2 of coffee worth P_1 and P_2 cents per pound, respectively, that will make a mixture of M lb. worth P_3 cents per pound.

12. Find the amounts of money M_1 and M_2 invested at the rates R_1 and R_2, respectively, when the total amount invested is T and the total interest per annum is I (dollars).

13. Find the speeds S_1 and S_2 of two automobiles that start from the same point in opposite directions and are a distance D miles apart in a time T hr., if one automobile travels twice as fast as the other. NOTE: For each automobile the distance traveled is the speed times the time.

14. Find the speeds S_1 and S_2 of two automobiles that start from the same point and are a distance D apart in a time T hr., if one automobile travels M m.p.h. faster than the other.

15. Find the amounts of money M_1 and M_2 invested at the rates R_1 and R_2, respectively, when the difference between the two amounts is D and the interest is I.

REVIEW QUESTIONS

1. Why are equations so useful in solving problems?
2. How are equations obtained in solving problems?
3. How many equations are needed in solving a problem?
4. When is it best to write equations that can be solved for a formula instead of a numerical result?
5. How does one obtain equations that lead to a formula instead of a numerical result?

CHAPTER 3

REVIEW OF FUNDAMENTAL OPERATIONS

The student who is to understand algebra and its uses must achieve three objectives: (1) cultivate the mental skills described in Chap. I, (2) learn to translate practical problems into equations, and (3) be able to solve equations and perform the other routine operations of algebra. In solving equations, it is often necessary to go through a considerable number of steps involving the fundamental operations of algebra: multiplication, division, addition, and subtraction. Before taking up the solution of equations, therefore, it is best to go through a systematic review of the fundamental operations that will be used later in solving equations. It is suggested that the student be as painstaking and industrious as possible in covering this review chapter, which is a necessary preparation for the more interesting material that follows.

10. General Numbers. In arithmetic, numbers are represented by the symbols 1, 2, 3, 4, etc., each of which has a definite value. In algebra, some numbers (called *general* numbers) are represented by letters of the alphabet, which can have any numerical values. They are used in two ways: (1) as *arbitrary* numbers, letters which in a specific equation can be assigned any desired value; and (2) as *unknowns*, letters which in a specific problem have definite numerical values *that are to be found.* For example, consider the formula for the area of a rectangle, $A = lw$. Here, A is the area of the rectangle, l is its length, and w is its width. If one wishes to determine the area of the rectangle, he assigns to l and w the values of the length and width of *that particular rectangle.* When this is done, A is determined; *i.e.*, it has a definite numerical value that is to be found. Observe that l and w are *arbitrary* numbers, whose values are assigned by choice, and that A is the *unknown*, whose value is to be found.

11. Fundamental Operations. The four fundamental operations of algebra are addition, subtraction, multiplication, and division.

12. Algebraic Expressions. A combination of numbers with the fundamental operations is called an *algebraic expression*. The parts of an expression that are separated by the signs + or − are called *terms*.

Example: $3x - 4y + 3z$ is an algebraic expression that consists of three terms.

13. Monomials and Multinomials. An algebraic expression consisting of only one term (such as $2x$) is called a *monomial;* an expression consisting of two terms ($2x - 3y$) is called a *binomial;* and an expression consisting of three terms ($2x - 3y + 4z$) is called a *trinomial.* In general, expressions containing more than one term are called multinomials; thus binomials and trinomials are types of multinomials.

14. Factors and Coefficients. The quantities multiplied together to form a product are called its *factors*. Thus 3, x, and y are the factors of the product $3xy$. Any factor is the *coefficient* of the remaining factors. In the product $3xy$, 3 is the coefficient of xy, x is the coefficient of $3y$, $3x$ is the coefficient of y, etc. In a less general sense, the term *coefficient* is used to mean only the numerical coefficient (3, in this case), and any other coefficient is referred to as a *literal* coefficient, since it involves letters. If the (numerical) coefficient is not indicated, it is understood to be 1.

15. Powers and Exponents. Consider the product

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

In order to express this number as conveniently as possible, the common practice is to write it as 2^6, the number 6 indicating the number of 2's that are multiplied together. Numbers like 2^6, 2^8, etc., are called *powers* of 2, and the number written at the upper right (to indicate how many times 2 is used as a factor) is called an *exponent*. The same system is employed with general numbers:

$a \cdot a = a^2$, usually read "a square" or "a squared."

$a \cdot a \cdot a = a^3$, usually read "a cube" or "a cubed."

$a \cdot a \cdot a \cdot a = a^4$, usually read "$a$ to the fourth power," or "a fourth," and so on. The symbol a means a^1, or a to the first power. Thus, when the power of a quantity is not shown explicitly, it is understood to be 1.

Example 1. Find the value of $3a^4$ when $a = 2$.

Solution: $3a^4 = 3 \cdot a \cdot a \cdot a \cdot a = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 48$

Example 2. If $x = 2$, $y = 5$, $z = 3$, evaluate the expression

$$x^2y + yz^3 - x^4z.$$

Solution:

$$x^2y = 2^2 \cdot 5 = 2 \cdot 2 \cdot 5 = 20$$
$$yz^3 = 5 \cdot 3^3 = 5 \cdot 3 \cdot 3 \cdot 3 = 135$$
$$x^4z = 2^4 \cdot 3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 48$$

Hence $x^2y + yz^3 - x^4z = 20 + 135 - 48 = 107$

EXERCISES (15)

1. Give the (numerical) coefficient of each of the following terms and the exponent of each letter. Also read each term aloud.

(*a*) $20a^6$ (*b*) $2a^3b^2$ (*c*) $\frac{1}{2}mv^2$ (*d*) $3uv^2w^3$ (*e*) $8xy^4z^7$

2. Use exponents to write the following as simply as possible:

(*a*) $xxxxx$ (*b*) aaa (*c*) $10bbbc$ (*d*) $3 \cdot 3 \cdot 3mmnnn$ (*e*) $8bbabba$

3. Determine the value of each of the following expressions when $x = 2$ and $y = 3$:

x^2, y^3, $6xy$, $3x^2y$, $x + y$, $2(x + 3y)$, $x^2 + y^2$, $3x^2 + 5y$

4. Evaluate each of the following expressions when $u = 1$, $v = 2$, $w = 3$, and $x = 4$:

w^2v^3, $5v^2 + 7$, $3u^5 + 2w^2$, $uvwx$, $u^2 + w^2 + v^2 + x^2$, $u + v - w + x$, $vw - ux$, $(u + v)(w + x)$, $3u^{16}$, w^4, $u^3 + v^3 + w^3$, $(w - u)^2$, $\frac{2}{3}w^2uv$, $\dfrac{w - u}{w - v}$, $(uvw)^2$, $uv + vw + wx$, $x + 3(u + v)^2$, $(x - w + 1)^5$, $(2vw)^2 - (ux)^2$, $w^3 - \frac{1}{4}x^2$

16. Numbers with Signs. In arithmetic, numbers have no signs. When we write

$$16 + 7 - 5 + 3 - 8$$

we use the signs + and − to indicate which of the numbers 16, 7, 5, 3, and 8 are to be added and which are to be subtracted. The signs, however, are not parts of the numbers. Thus (in arithmetic) there are no positive or negative numbers; there are just *numbers*, of which some are whole numbers and some are fractions.

In algebra, each number includes a sign, + or −, as a part of itself. Thus, +5 is an *additive* (or positive) number, and −5 is a

subtractive (or negative) number. The use of the sign as a part of the number makes it possible to let the number *carry with itself* the quality of being additive or subtractive.

If you were weighing bales of cotton, and if you wished to mark on each bale an indication of how much its weight was "off" from the prescribed figure of 500 lb., how would you mark a bale weighing 510 lb.? How would you mark a bale weighing 490 lb.?

In general, negative numbers are used to represent the opposite of whatever is represented by a positive number. For example, if profit is recorded as positive, loss can be recorded as negative. Consider the following business record:

Jan.	\$500 profit	+ \$500
Feb.	2,000 profit	+2,000
Mar.	100 loss	− 100
Apr.	2,500 profit	+2,500
May	750 loss	− 750
June	250 loss	− 250
July	1,000 profit	+1,000
Aug.	200 profit	+ 200
Sept.	1,600 profit	+1,600
Oct.	1,500 loss	−1,500
Nov.	600 profit	+ 600
Dec.	400 profit	+ 400

Observe that the signs + and − convey the same information as the words *profit* and *loss*. An alternative procedure would be to record losses in red, to distinguish them from profits.

When distance in one direction is considered positive, distance in the other direction can be represented as negative, as shown below:

−5 −4 −3 −2 −1 0 1 2 3 4 5

On this scale, distance to the right of zero is positive, and distance to the left is negative. Thus, the number 3 is represented by the distance from 0 to 3, on this scale.

A negative number is the result of subtracting a positive number from zero. On the above scale, if we start at zero and move 3 units *to the left*, we reach −3. Thus,

1. Subtracting a positive number from zero gives a negative number of the same size, *viz.*,

	0	0	0
Subtract	25	17	33
Result	−25	−17	−33

2. Subtracting a negative number from zero gives a positive number of the same size:

	0	0	0
Subtract	−21	−38	−56
Result	+21	+38	+56

Example 1. If the temperature at noon was 24° (above zero) and if it dropped 25° between noon and 6 P.M., what was the temperature at the latter time?

Solution: In order to reach zero, the temperature had to drop 24°. In all, however, it dropped 25°, thus reaching a temperature of 1° below zero, or −1°, at 6 P.M. If it should drop 4° more, the temperature would be 5° below zero, or −5°.

17. Addition and Subtraction of Signed Numbers. In adding and subtracting signed numbers, try to gain the feeling that you understand what you are doing, rather than memorizing formal rules. Study each of the following, but *do not memorize the rules.*

1. Numbers of the same sign may be combined directly in addition, keeping the same sign:

75	−24	−17
52	−37	−26
17	−61	−33
144		−76

2. In subtracting a number from a larger number of the same sign, subtract directly, retaining the same sign:

	73	−84	−117
Subtract	27	−25	− 21
	46	−59	− 96

3. In subtracting a number from a smaller number of the same sign, make use of the method illustrated by the following examples:

Example 1. Subtract 75 from 50.

Solution: Observe that we cannot take away 75 from 50, just as a man who has only \$50 in the bank cannot withdraw \$75. If we take away as much as we can, or 50, we still have 25 *to subtract from zero. The result of subtracting* 25 *from zero* is -25; hence $50 - 75 = -25$.

The first step in such a subtraction is to observe that the result will be negative; the second step is to evaluate the difference between the two numbers by subtracting the *smaller* from the *larger*. Check the following:

	33	27	44
Subtract	64	41	63
Result	-31	-14	-19

Example 2. Subtract -95 from -32.

Solution: The result of this subtraction will be positive, because part of -95 will have to be subtracted from zero. The difference between 95 and 32 is 63; hence the answer is $+63$.

Check the following:

	-27	-58	-123
Subtract	-44	-93	-249
	$+17$	$+35$	$+126$

Numbers can be added or subtracted directly only if their signs are alike, as in the preceding cases. When their signs are unlike, use your understanding of the "oppositeness" of positive and negative numbers to obtain numbers of like sign. In doing this, *mentally* change the sign of the number to be added or subtracted, and at the same time change from addition to subtraction, or vice versa. The method will now be illustrated.

4. Subtracting a negative number is equivalent to adding a positive number of the same size.

	9	23	96
Subtract	-2	-4	-23
Result	11	27	119

In the first of these examples, observe that what we do is equivalent to subtracting -2 from zero (instead of 9) and then adding the result $(+2)$ to 9. Instead of *subtracting* -2, we *add* 2.

5. Subtracting a positive number is equivalent to adding a negative number:

	−24	−35	−59
Subtract	5	17	65
Result	−29	−52	−124
	−33	−57	−545
Subtract	7	14	322
Result	−40	−71	−867

In each of the examples shown, we *add* a *negative* number instead of subtracting a positive number.

6. In adding two numbers of unlike signs, subtract the smaller from the larger and prefix the sign of the larger. In doing this, *first* record the sign, then perform the subtraction:

	37	−45	−58
Add	−12	32	76
Result	25	−13	18
	5	−43	35
Add	−14	15	−72
Result	−9	−28	−37

7. In adding a group of numbers, of which some are positiv and some negative, combine positive and negative numbers separately; then combine the results:

Example 3. Add 32, −17, −24, and 76.

Solution: Adding positive and negative numbers separately and combining the results,

Positive	Negative		
32	−17	Positive	108
76	−24	Negative	−41
108	−41	Total	67

It is not necessary to write out everything, since the positive and negative subtotals can be obtained mentally. Thus, mentally add 32

and 76, writing 108 (as above). Mentally, add −17 and −24, writing −41. Then add 108 and −41, obtaining 67.

EXERCISES (17)

Add the following:

1. −84 −12	2. −17 −54	3. 65 −12	4. −23 85
5. 55 −82	6. 75 −62	7. −56 35	8. 32 −61
9. 33 −71	10. 148 −61	11. −245 79	12. −140 −225
13. −27 −26 −54 67	14. −242 65 47 84	15. −52 −16 95 −11	16. −16 −24 −72 385
17. 427 −116 −94 162	18. 965 −25 −72 −34	19. −75 −62 154 −12	20. −60 −71 −42 17

Subtract the second number from the first:

21. −24 −9	22. −36 −11	23. −31 −45	24. −74 −91
25. 8 24	26. 16 43	27. 18 67	28. 17 44
29. −9 −25	30. −12 −33	31. −64 −92	32. −120 −260
33. 16 −4	34. 9 −7	35. 23 −62	36. 24 −71
37. −28 5	38. −32 16	39. −85 46	40. −102 237

18. Multiplication and Division of Signed Numbers. When two or more numbers are combined in multiplication or division, the negative signs of individual factors may be removed and applied to the entire quantity. Thus,

$$(-2) \times 3 = -(2 \times 3) = -6$$
$$3 \times (-2) = -(3 \times 2) = -6$$
$$(-2) \times (-3) = - -(3 \times 2) = - -6 = +6$$

In the third case, there are two negative signs; therefore the result is positive. In general, an *even* number of negative signs results in a positive sign; while an *odd* number of negative signs results in a negative sign.

Examples:

$$(3)(-5) = -15$$
$$(-2)(-4) = +8$$
$$(-4)(6)(-7) = +168$$
$$(-3)(4)(-5)(-7) = -420$$

> ***In combining any number of quantities by multiplication and division, if the number of negative quantities is odd, the result is negative; otherwise, it is positive.***

EXERCISES (18)

Multiply:

1. $(-5)(-8)$
2. $(-6)(-8)$
3. $(-24)(3)$
4. $(3)(-21)$
5. $(2)(-3)(-2)$
6. $(-2)(-3)(-4)$
7. $(-7)(-3)(5)$
8. $(7)(-3)(2)(-4)$
9. $(2)(3)(4)(-3)$
10. $(2)(-5)(-6)(-2)$
11. $(3)(-2)(-3)(-5)$
12. $(-1)(-2)(-3)(-4)$

Divide:

13. $(6) \div (-2)$
14. $(-6) \div (-2)$
15. $(-14) \div (7)$
16. $\dfrac{87}{-3}$
17. $\dfrac{-87}{-3}$
18. $\dfrac{-69}{23}$

Evaluate:

19. $\dfrac{(2)(-10)}{4}$
20. $\dfrac{(-9)(-6)}{-27}$
21. $\dfrac{-72}{(-6)(2)}$

22. $\dfrac{60}{(-3)(5)}$ 23. $\dfrac{(9)(-8)}{(3)(-12)}$ 24. $\dfrac{(-6)(-10)}{(-4)(5)}$

25. $\dfrac{48}{(-6)(-4)}$ 26. $\dfrac{(-8)(-6)}{(-3)(-4)}$ 27. $\dfrac{(-9)(-2)}{(-6)(3)}$

19. Order of Operations. In algebra the order in which operations are to be performed is usually indicated by grouping symbols: parentheses, brackets, or fraction bars. Whenever the order is not indicated, multiplication and division are performed first, since they occur within terms; and addition and subtraction, which combine terms, are completed afterward:

Example 1. Evaluate $(-3)(-4) - (-5)(+2)$.

Solution: $(-3)(-4) = +12$ and $(-5)(+2) = -10$

Then $+12 - (-10) = +22$ is the desired result.

Example 2. Evaluate $[-4 + 7][-5 - (-9)]$.

Solution: $-4 + 7 = 3$ and $-5 - (-9) = 4$

Then $3 \times 4 = 12$

Example 3. Evaluate $\dfrac{-5 - (-13)}{-2} - \dfrac{-12(+3)}{4 - (-2)}$.

Solution: $$\frac{-5 - (-13)}{-2} = \frac{-5 + 13}{-2} = \frac{8}{-2} = -4$$

and $$\frac{-12(+3)}{4 - (-2)} = \frac{-36}{6} = -6$$

Then $$-4 - (-6) = -4 + 6 = 2$$

EXERCISES (19)

Evaluate:

1. $5(-7) - 2(-3)$
2. $-3(-5) + 2(-4)$
3. $2(-6) - 3(-4)$
4. $6 - 3(-1) - 5(-2)$
5. $\dfrac{-6}{2} - \dfrac{-12}{3}$
6. $\dfrac{18}{-3} - \dfrac{-15}{-5}$
7. $2(-3)^2 - 5(-2)^3$
8. $\dfrac{-4}{2} - \dfrac{-26}{-13}$
9. $\dfrac{32}{-4} - \dfrac{18}{-3}$
10. $\dfrac{24}{-3} - \dfrac{-24}{6}$
11. $\dfrac{5 - (-5)}{(-2) + (-3)}$
12. $\dfrac{22 - (-14)}{-16 - (-7)}$
13. $\dfrac{-31 - (-19)}{13 + (-1)}$
14. $\dfrac{-23 - (-2)}{-12 - (-54)}$
15. $\dfrac{16 - (-23)}{-7 - (-20)}$

20. Addition and Subtraction of Algebraic Expressions.

Just as	$\$10 + \$15 =$	$\$25$,	so does	$10a + 15a =$	$25a$
Just as	$\$42 - \$27 =$	$\$15$,	so does	$42a - 27a =$	$15a$
Just as	$\$16 - \$35 =$	$-\$19$,	so does	$16a - 35a =$	$-19a$

Terms whose literal parts are alike are called *similar*, or *like*, terms. Examples of similar terms are $6a$, $5a$, $17a$, $-24a$, or $16x^2y$, $-3x^2y$, $-17x^2y$. They can be added or subtracted by adding or subtracting their numerical coefficients and taking the result as the coefficient of the common literal part.

Example 1. Simplify $6a + 23a - 17a + 2a$ by combining similar terms.

Solution: $6a + 23a - 17a + 2a = (6 + 23 - 17 + 2)a = 14a$

Example 2. Simplify $26xy^3z + 3xy^3z - 12xy^3z$.

Solution: $26xy^3z + 3xy^3z - 12xy^3z = (26 + 3 - 12)xy^3z = 17xy^3z$

Example 3. Simplify $6a^2 - 2a^2 + 2b$.

Solution: $6a^2 - 2a^2 + 2b = (6 - 2)a^2 + 2b = 4a^2 + 2b$

NOTE: $2b$ cannot be added to $4a^2$, since they are not like terms.

One cannot add terms that are unlike, just as one cannot add 6 oranges and 5 apples, or $6a$ and $5b$. One can write, "The sum of $6a$ and $5b$ is $6a + 5b$," thereby *indicating* the addition, but not actually performing it.

In order to add a series of monomials, one collects the similar terms into groups that can be combined into single terms.

Example 4. Simplify $5a^2b + 4z + 6x + 2z - 3x - 2a^2b$.

Solution: Rearranging, one obtains the expression

$$(5a^2b - 2a^2b) + (4z + 2z) + (6x - 3x)$$

which is equivalent to $3a^2b + 6z + 3x$.

In adding multinomials, arrange the expressions in rows in such a way that the similar terms are in the same vertical columns; then add each column separately.

Example 5. Add $2x - 3y + 4z + 3z^2$ and $2z^2 + 5y + 7x$.

Solution:

$$\begin{array}{l} 2x - 3y + 4z + 3z^2 \\ \underline{7x + 5y \qquad + 2z^2} \\ 9x + 2y + 4z + 5z^2 \end{array}$$

Example 6. Add:

$$\begin{array}{l} 6x^2 - 3y + 7x - 17z \\ 12y - 2x^2 + 3z \\ 5x - 2z + 4x^2 - 6y \end{array}$$

$$
\begin{array}{lrrrr}
\text{Rearranging,} & 6x^2 - & 3y + & 7x - & 17z \\
& -2x^2 + & 12y & + & 3z \\
& 4x^2 - & 6y + & 5x - & 2z \\
\hline
\text{Adding,} & 8x^2 + & 3y + & 12x - & 16z
\end{array}
$$

In subtracting, also, it is convenient to arrange similar terms in the same vertical columns.

Example 7. Subtract $6x^2 - 3y + 5x - 17z$ from $7x - 2z + 8x^2 - 12y$.

$$
\begin{array}{lrrrr}
\text{Rearranging,} & 7x - & 2z + & 8x^2 - & 12y \\
& 5x - & 17z + & 6x^2 - & 3y \\
\hline
\text{Subtracting,} & 2x + & 15z + & 2x^2 - & 9y
\end{array}
$$

Example 8. Subtract: $64x^2y - 25y \quad + 16x$
$\qquad 62y \quad - 25x^2y - 32x^2$

$$
\begin{array}{lrrrr}
\text{Rearranging,} & 64x^2y - & 25y + & 16x & \\
& -25x^2y + & 62y & & - 32x^2 \\
\hline
& 89x^2y - & 87y + & 16x + & 32x^2
\end{array}
$$

[Note that $0 - (-32x^2) = +32x^2$.]

EXERCISES (20)

Add:

1. $6a, 2a, -7a, 13a$
2. $6a + 3b, 7a - 3b, a + b$
3. $2ax, -4ax, ax, 6ax$
4. $2a + 3b - 6c, 3a - 8b + 7c, 8a - 7b + 4c$
5. $7a + 6c - 3b, 12b + 6c - 13a, 2c + 15a$
6. $6ay + 2bx, 4bx - 2ay, 5ay - 5bx$
7. $3x^2 - 2xy, 42x^2 - 3xy, 16xy, -30x^2$
8. $a^2 + b^2 + c^2, 2a^2 - 3b^2, 2c^2 - 3b^2$
9. $a^2 + 4a, -2a + 7$
10. $3a^2 - 4ab + 6b^2, 2ab - 2a^2 + 7b^2$
11. $x^2 + xy, -xy + y^2$
12. $a^2 + ab + b^2, -ab + 3b^2, a^2 - 2b^2$
13. $x^2 + 8x - 6, x^2 - 2x + 3, x^2 + y^2$

Subtract the second expression from the first:

14. $5a - 3b; 2a + 7b$
15. $6x + 3y; 2x - 5y$
16. $6x^2 + y^2; 2x^2 - 3y^2$
17. $4a^2 + 3b^2; 6b^2 - 2a^2$
18. $7mn^2 + 8m^2n; 2m^2n - 3mn^2$
19. $5a - 2b + c; 3a - b - c$
20. $8x^2 - 10y^2 - 2z^2; -5x^2 - 9y^2 + 2z^2$

21. $8m^2 - 6mn - 2$; $6m^2 - 4mn + 4$
22. $4x^4 - 4x^3 - 2x + 1$; $2x^4 - 3x^3 - x^2$
23. $10a^3 - 7a + 13$, $3a^3 - 2a^2 + 15$
24. $-2x^3 - 5x^2 + 7x - 9$, $7x^3 + 8x - 6$
25. $\frac{1}{2}x + \frac{1}{3}y$, $\frac{1}{3}x - \frac{1}{2}y$
26. $\frac{1}{6}a^2 - \frac{1}{2}b^2$, $\frac{2}{3}a^2 - b^2$
27. $6.21a^2 + 3.27ab - 5.75b^2$, $9.37a^2 - 5.83b^2$

21. Parentheses. Parentheses are used to indicate that the terms within them are grouped together as a single quantity. Brackets [ ] and braces { } are used in the same way, usually to avoid placing parentheses within parentheses. A factor placed outside a set of parentheses is a factor of the entire expression within the parenthesis; hence every term inside must be multiplied by the factor in order to remove the parentheses. For example, $2(x + 2y - 3z) = 2x + 4y - 6z$, and

$$-3(2x - 3y + 4z) = -6x + 9y - 12z.$$

A minus sign outside a set of parentheses indicates that every term inside is to be multiplied by -1, or that all the signs within must be changed in order to remove the parentheses.

Example 1. $+(3x - 4y + z) = 3x - 4y + z$
but $-(3x - 4y + z) = -3x + 4y - z$

In inserting parentheses, one must observe the same convention.

Example 2. $2x + 3y - z = +(2x + 3y - z)$
but $-5x - 4y + z = -(5x + 4y - z)$

Any factor common to all the terms of an expression can be placed outside the parentheses enclosing the expression, provided each term inside is divided by the common factor.

Rules for Parentheses

1. Each term within parentheses must be multiplied by the factor outside the parentheses before the parentheses can be removed. Note that this includes cases where the parentheses are preceded by the sign —.

2. When enclosing an expression in parentheses and placing a factor outside, one must divide every term in the expression by the factor placed outside. This includes the case in which the factor placed outside is —1.

EXERCISES (21)

Simplify, by removing the parentheses and combining similar terms:

1. $(3x - y) + (2x - 5y)$
2. $(4x - 7) - (5x + 6)$
3. $(2a - 3b) - (6a - 4b)$
4. $(2a - 1) - (4a + 3)$
5. $(a + b - c) + (a - b + c)$
6. $(a - b + c) - (a + b - c)$
7. $(5x^2 - 9x + 1) - (3x^2 - 5x - 2)$
8. $-2(3a^2 - 5a - 6b) - 3(2a^2 - 4a + 2b)$
9. $(4a - b) - (3a + 4b) - (6a - 5b)$
10. $3x^2 - (x^2 - 2) + 2(x^2 - 3x + 7)$
11. $4x^2 - 17x + (2x^2 - 4) - 3(5x^2 - 4x + 3)$
12. $7x^2 - 14y^2 - 7(x^2 - y^2) + 2(4x^2 + 3y^2 + 6xy)$

22. Multiplication of Monomials. The multiplication of monomials will be illustrated by examples:

Example 1. $3a^2 \times 4a^3 = 12a^5$

Proof: Since $3a^2 = 3 \cdot a \cdot a$ and $4a^3 = 4 \cdot a \cdot a \cdot a$
then $3a^2 \times 4a^3 = 3 \cdot a \cdot a \cdot 4 \cdot a \cdot a \cdot a = 12a \cdot a \cdot a \cdot a \cdot a = 12a^5$

The coefficient of the product of two monomials is obtained by multiplying the coefficients of the two monomials, and the exponent of each letter in the product is the sum of its exponents in the monomials. In multiplying monomials containing powers of several letters, it must be remembered that the exponents of one letter must not be combined with those of another.

Example 2. $12a^2b^3c \times 5a^3b^4c^2 = 60a^5b^7c^3$
(Note that $c \times c^2 = c^3$, since $c = c^1$.)

Literal exponents are treated just like numerical ones, except that their addition can only be indicated, not actually performed.

Example 3. $3x^3y^4 \times 4x^my^n = 12x^{3+m}y^{4+n}$

In multiplying two or more monomials, multiply the coefficients together to obtain the coefficient of the product, and include in the product all the factors that appear in any of the monomials, using as the exponent of each letter the sum of its exponents in the separate monomials.

EXERCISES (22)

Multiply:

1. $\begin{array}{r} 3x \\ \underline{4} \end{array}$ 2. $\begin{array}{r} -3a \\ \underline{-2} \end{array}$ 3. $\begin{array}{r} 3x \\ \underline{2x} \end{array}$ 4. $\begin{array}{r} 5a \\ \underline{-3a} \end{array}$

5. $\begin{array}{r} -6ab \\ \underline{6b} \end{array}$ 6. $\begin{array}{r} 4x^2y \\ \underline{-2xy} \end{array}$ 7. $\begin{array}{r} 5a^2bc^2 \\ \underline{20ab^4c^3} \end{array}$ 8. $\begin{array}{r} 2ab^2 \\ \underline{3ba^2} \end{array}$

9. $(-2x^3y^7z^2)(3y^2z^5x^2)$
10. $(5xy)(-6x^2yz^2)$
11. $(4ab^2)(-2a^2bc)(-3b^3c^3)$
12. $(6ac)(-8ab)(-2ab)(-2c)$
13. $(6x^2y^3)(4x^{a+3}y^{b+1})$
14. $(3a^3b^2c^5)(a^mb^nc^3)$

23. Multiplication by Monomials.

> ***The product of a multinomial and a monomial is the algebraic sum of the product terms formed by multiplying each term of the multinomial by the monomial.***

Example 1. Multiply $3x + 2$ by $2x^2$.
Solution: $2x^2(3x + 2) = (2x^2)(3x) + (2x^2)(2) = 6x^3 + 4x^2$

It is usually more convenient to multiply in vertical order, *viz.*,

$$\begin{array}{l} 3x + 2 \\ \underline{2x^2} \\ 6x^3 + 4x^2 \end{array} \qquad \begin{array}{l} 12a^2b + 3ab^2 \\ \underline{2a^2} \\ 24a^4b + 6a^3b^2 \end{array} \qquad \begin{array}{l} -3x^2y^2 + 4xy^3 \\ \underline{-2xy} \\ +6x^3y^3 - 8x^2y^4 \end{array}$$

Example 2. Multiply $x^n + y^m$ by x^2y^3.

$$x^2y^3(x^n + y^m) = x^2 \cdot x^n \cdot y^3 + x^2 \cdot y^3 \cdot y^m = x^{2+n}y^3 + x^2y^{3+m}$$

EXERCISES (23)

Multiply as indicated:

1. $\begin{array}{r} 4x - 3y \\ \underline{5} \end{array}$ 2. $\begin{array}{r} 3a - 2b \\ \underline{4} \end{array}$ 3. $\begin{array}{r} 7a + 6 \\ \underline{-3} \end{array}$

4. $\begin{array}{r} -3x + 4y \\ \underline{2xy} \end{array}$ 5. $\begin{array}{r} 3ab + 4b^2 \\ \underline{2a^2b} \end{array}$ 6. $\begin{array}{r} 5m + 4n - 3mn \\ \underline{2mn^2} \end{array}$

7. $(-3y^2)(3x^2 - 2x^2 - 2y)$
8. $-3x^2(2xy - 3x^2 - y)$
9. $-5ab^2(3a^2 - 6b + 2ab - 7)$
10. $2x^2 \cdot x^n$
11. $(2x^2 + 3y^2)x^n$
12. $(x^m + y^n)x^ny^m$

24. Multiplication of Multinomials.

The product of two multinomials is the algebraic sum of all the product terms obtained by multiplying every term of one multinomial by each term of the other.

Example 1. Multiply $2a + 3$ by $3a - 2$.

$$\begin{aligned}(2a+3)(3a-2) &= (2a)(3a) + (2a)(-2) + 3(3a) + 3(-2)\\ &= 6a^2 - 4a + 9a - 6\\ &= 6a^2 + 5a - 6\end{aligned}$$

or, arranging the multinomials in vertical order for more convenient multiplication:

$$\begin{array}{rrr} 2a & +3 & \\ 3a & -2 & \\ \hline 6a^2 & +9a & \\ & -4a & -6 \\ \hline 6a^2 & +5a & -6 \end{array}$$

Example 2. Multiply $6a^2 - 5a + 7$ by $2a + 3$.

Solution:

$$\begin{array}{rrrr} 6a^2 & -5a & +7 & \\ 2a & +3 & & \\ \hline 12a^3 & -10a^2 & +14a & \\ & +18a^2 & -15a & +21 \\ \hline 12a^3 & +8a^2 & -a & +21 \end{array}$$

Notice that the vertical arrangement makes it convenient to place *like terms* in the same vertical columns, thus facilitating the combination of terms.

EXERCISES (24)

Multiply as indicated:

1. $\begin{array}{r} 4a - 3 \\ \underline{3a + 5} \end{array}$
2. $\begin{array}{r} 6x - 1 \\ \underline{2x - 5} \end{array}$
3. $\begin{array}{r} 4m - 2n \\ \underline{m - 3n} \end{array}$
4. $\begin{array}{r} 4y + 7 \\ \underline{2y - 3} \end{array}$
5. $\begin{array}{r} 6x^2 - 2y^2 \\ \underline{2x^2 - y^2} \end{array}$
6. $\begin{array}{r} m^2 - 3n \\ \underline{2m^2 + n} \end{array}$
7. $\begin{array}{r} -m^2 + 5 \\ mn - 7n \end{array}$
8. $\begin{array}{r} a^2 - 2b^2 \\ 2b^2 + 3a \end{array}$
9. $\begin{array}{r} 7m - 2n \\ 4m^2 - 3n \end{array}$

10. $\begin{array}{r} 6a - 3b \\ \underline{3b + 2a} \end{array}$

11. $\begin{array}{r} 6a - 5b + 7 \\ \underline{2a - b} \end{array}$

12. $\begin{array}{r} a^2 + ab + b^2 \\ \underline{2a + b} \end{array}$

13. $\begin{array}{r} 3x - 7y + 1 \\ \underline{2x - 8y} \end{array}$

14. $\begin{array}{r} 6m^2 - 3mn - 5n^2 \\ \underline{-m^2 - 2mn + 2n^2} \end{array}$

15. $\begin{array}{r} 3x^3 - x^2 + 2x - 4 \\ \underline{5x^2 - x + 3} \end{array}$

25. Division of Monomials.

Example 1. $6x^5 \div 2x^2 = \dfrac{6x^5}{2x^2} = 3x^3$

Proof: Since $6x^5 = 6xxxxx$, and $2x^2 = 2xx$

$$\frac{6x^5}{2x^2} = \frac{6xxxxx}{2xx} = 3xxx = 3x^3$$

> ***The coefficient of the quotient of two monomials is the quotient of their coefficients, and the exponent of each letter in the quotient is its exponent in the dividend, minus its exponent in the divisor.***

Example 2. Divide $-64a^4b^3c$ by $-4a^2b$.

Solution: $\dfrac{-64a^4b^3c}{-4a^2b} = \dfrac{-64}{-4} \cdot \dfrac{a^4}{a^2} \cdot \dfrac{b^3}{b} \cdot c = 16a^{4-2}b^{3-1}c = 16a^2b^2c$

Example 3. Divide $-33x^7y^5z^3$ by $11x^3y^2z^3$.

Solution: $\dfrac{-33x^7y^5z^3}{11x^3y^2z^3} = -3x^{7-3}y^{5-2}z^{3-3} = -3x^4y^3z^0 = -3x^4y^3$

Note that $\dfrac{z^3}{z^3} = z^{3-3} = z^0$. We know $\dfrac{z^3}{z^3} = 1$; hence $z^0 = 1$. Any number raised to the zero power is equal to unity.

Literal exponents should be treated just like numerical ones, except that the subtraction can be only indicated, not completed.

Example 4. $6x^my^n \div 3x^2y^3 = \dfrac{6x^my^n}{3x^2y^3} = 2x^{m-2}y^{n-3}$

> ***In dividing monomials, first divide their coefficients, writing the result as the new coefficient. Then obtain the exponent of each letter in the quotient, by subtracting its exponent in the divisor from its exponent in the dividend.***

EXERCISES (25)

Divide as indicated:

1. $24x^3 \div (-3)$
2. $-27a^5 \div 3a^2$
3. $-63a^3b^4 \div (-9ab^2)$
4. $\dfrac{-81x^2y^3z^5}{3xy^2z^3}$
5. $\dfrac{87a^3b^3c^3}{-3abc}$
6. $\dfrac{-28m^5n^7}{-7m^3n^2}$
7. $\dfrac{4x^my^n}{2x^3y^2}$
8. $\dfrac{27x^ay^b}{3x^by^a}$
9. $\dfrac{6x^{m+4}y^{n+3}}{3x^2y^3}$
10. $\dfrac{12m^{a+3}n^{b-7}}{4n^{b+2}m^{a-1}}$
11. $\dfrac{24x^{m+3}y^{n-4}}{12x^{n+2}y^{m-5}}$
12. $\dfrac{33a^{2x-1}b^{y+2}c^{z-3}}{11a^{x+2}b^{y-1}c^{2z-4}}$

26. Division by Monomials.

Example 1. Divide $(24x^3 - 9y^4)$ by 3.

Solution: $\dfrac{24x^3 - 9y^4}{3} = \dfrac{24x^3}{3} - \dfrac{9y^4}{3} = 8x^3 - 3y^4$

Example 2. $\dfrac{32x^5y^3 - 12x^3y^2 + 26x^2y^3}{-2x^2y^2} = -16x^3y + 6x - 13y$

In dividing a multinomial by a monomial, divide each term of the multinomial by the monomial, and write the algebraic sum of the quotient terms thus obtained.

EXERCISES (26)

1. $(12a^4 - 24a^3) \div 6a^2$
2. $(25m^4 - 30m^2n^2) \div 5m^2$
3. $\dfrac{12a^2b^3 - 21a^3b^2}{-3a^2b^2}$
4. $\dfrac{8x^7y^5z^6 - 4x^6y^4z^3}{-4x^4y^2z^3}$
5. $(7a^2b^2c^3 - 14ab^3c^2 - 7bc^3) \div 7bc$
6. $(xyz^2 - xy^2z + x^2yz) \div (-xyz)$
7. $(21a^2b^2c^2 - 28a^3b^2c - 63ab^2c^3) \div (-7abc)$
8. $(-18x^3y^4 - 30x^4y^3 - 36x^3y^2) \div (-6x^2y^2)$
9. $(6.6a^2b^2 - 1.5a^3b + 2.7a^3b^3) \div (-.3a^2b)$
10. $(a^2 - \frac{1}{5}ab + ab^2) \div (-\frac{1}{8})$
11. $(1.6x^5 - 3.4x^4 - 5.4x^3) \div .2x^2$
12. $(14x^5y^2 - 28x^3y^7) \div 7x^2y^m$

27. Division of Multinomials. By definition, division is the process of finding an expression that can be multiplied by the divisor to obtain the dividend. Division, then, is the inverse of multiplication. For example,

$$(3a - 2)(2a + 3) = 6a^2 + 5a - 6$$

Thus
$$\frac{6a^2 + 5a - 6}{3a - 2} = 2a + 3$$

and also
$$\frac{6a^2 + 5a - 6}{2a + 3} = 3a - 2$$

In order to determine $\frac{6a^2 + 5a - 6}{3a - 2}$ directly, *i.e.*, without recourse to the first equation, one should arrange the work as follows:

Example 1. Divide $6a^2 + 5a - 6$ by $3a - 2$.

$$\begin{array}{rrr|l} \text{Dividend} \rightarrow 6a^2 & + 5a & - 6 & 3a - 2 \leftarrow \text{Divisor} \\ \cline{4-4} 6a^2 & - 4a & & 2a + 3 \leftarrow \text{Quotient} \\ \cline{1-2} & 9a & - 6 & \\ & 9a & - 6 & \\ \cline{2-3} \end{array}$$

Procedure:

1. Divide the first term ($6a^2$) of the dividend by the first term ($3a$) of the divisor, obtaining $2a$, the first term of the quotient. Write this in the place shown.
2. Multiply the entire divisor by $2a$, writing the result under the dividend *with similar terms in the same vertical columns.*
3. Subtract to obtain the remaining portion of the dividend, $9a - 6$.
4. Divide the first term ($9a$) of the new dividend by the first term ($3a$) of the divisor, obtaining the next term ($+3$) of the quotient.
5. Write this in its proper place and multiply the entire divisor by it, writing the result under the dividend and subtracting as before.
6. Continue the process until the remainder of the dividend is either zero, or of lower degree (in the letter of arrangement) than the divisor.

The degree of a term is the sum of the exponents of all the letters it contains. Thus $6x^3y^5$ is of eighth degree and $5x^2y^4$ is of sixth degree. The degree of a multinominal is the degree of its term of highest degree. Thus, $6x^3y^5 + 5x^2y^4$ is a binomial of the eighth degree. However, the degree of a term *in a particular letter* is the exponent of *that letter.* Hence $6x^3y^5$ is of third degree in x and of fifth degree in y.

Example 2. Divide $4x^4 - 9x^2y^2 - 3y^4 - 8x^3y + 16xy^3$ by

$$xy + 2x^2 - 3y^2$$

NOTE: It is necessary always to arrange the dividend and divisor in descending powers of some letter, called the *letter of arrangement,* and to leave blank spaces for any powers of this letter that do not appear in the dividend. The letter of arrangement in this problem could be either x or y. Check the division process in detail, as follows:

$$
\begin{array}{l|l}
4x^4 - 8x^3y - 9x^2y^2 + 16xy^3 - 3y^4 & 2x^2 + xy - 3y^2 \\
\underline{4x^4 + 2x^3y - 6x^2y^2} & \overline{2x^2 - 5xy + y^2} \\
\quad -10x^3y - 3x^2y^2 + 16xy^3 - 3y^4 & \\
\quad \underline{-10x^3y - 5x^2y^2 + 15xy^3} & \\
\qquad\qquad +2x^2y^2 + xy^3 - 3y^4 & \\
\qquad\qquad \underline{+2x^2y^2 + xy^3 - 3y^4} &
\end{array}
$$

Example 3. Divide $7a - 3a^2 - 7$ by $a - 3$.

Solution:

$$
\begin{array}{l|l}
-3a^2 + 7a - 7 & a - 3 \\
\underline{-3a^2 + 9a} & \overline{-3a - 2} \\
\qquad - 2a - 7 & \\
\qquad \underline{- 2a + 6} & \\
\qquad\qquad - 13 &
\end{array}
$$

The remainder -13 is of lower degree than $a - 3$; hence the division is complete and the quotient is $-3a - 2 - \frac{13}{a-3}$. One can always tell when to stop dividing by observing that further division produces a quotient term with letters in the denominator. In this example, if further division had been attempted, the next term in the quotient would have been $\frac{-13}{a}$.

Example 4. Divide $4x^4 - 3x^2 + 7x - 3$ by $x - 2$.

$$
\begin{array}{l|l}
4x^4 \;.\;.\;. - 3x^2 + 7x - 3 & x - 2 \\
\underline{4x^4 - 8x^3} & \overline{4x^3 + 8x^2 + 13x + 33} \\
\quad + 8x^3 - 3x^2 + 7x - 3 & \\
\quad \underline{+ 8x^3 - 16x^2} & \\
\qquad + 13x^2 + 7x - 3 & \\
\qquad \underline{13x^2 - 26x} & \\
\qquad\qquad + 33x - 3 & \\
\qquad\qquad \underline{33x - 66} & \\
\qquad\qquad\quad + 63 &
\end{array}
$$

Ans.: $4x^3 + 8x^2 + 13x + 33 + \frac{63}{x-2}$.

Use the following rule to learn the process of division, but do not attempt to memorize the rule itself.

In dividing a multinomial by a multinomial:

1. Arrange both dividend and divisor in descending powers of the chosen letter of arrangement. If any powers are missing, leave blank spaces for them.

2. Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

3. Multiply the entire divisor by the term obtained in (2), writing the result below the dividend with similar terms in vertical columns.

4. Subtract the result of (3) from the dividend to obtain the remaining portion of the dividend.

5. Divide the first term of the remainder (of the dividend) by the first term of the divisor to obtain the next term of the quotient.

6. Multiply the entire divisor by the term of (5) and subtract it from the remainder.

7. Continue the process described in (5) and (6) until the remainder is of lower degree than the divisor.

8. The result of the division is the quotient plus the fraction whose numerator is the remainder of (7) and whose denominator is the divisor.

EXERCISES (27)

Divide:

1. $x^2 + x - 6$ by $x + 3$
2. $12x^2 + 7xy - 12y^2$ by $4x - 3y$
3. $6a^2 - ab - 2b^2$ by $2a + b$
4. $8x^4 - 8x^2 - 6$ by $2x^2 - 3$
5. $3y^2 + 8y + 4$ by $3y + 2$
6. $12m^2 + 4mn - 5n^2$ by $2m - n$
7. $2xy^2 + x^3 + 9x^2y - 48y^3$ by $x - 2y$
8. $3x^3 - 5x^2 + 2$ by $x - 1$
9. $4y^3 - 3y + 1$ by $2y - 1$
10. $m^4 - 7m^3 + 8m^2 + 28m - 48$ by $m - 3$
11. $2x^4 - 3x^3y - 16xy^3 + 24y^4$ by $2x - 3y$
12. $24y^4 + y - 8y^3 + 1$ by $2y - 1$
13. $m^2 - 7m + 11$ by $m - 5$
14. $4m^4 - 9m^2n^2 - 3n^4 + 16mn^3 - 8m^3n$ by $-3n^2 + 2m^2 + mn$
15. $3x^2 - 7x + 14$ by $x - 2$
16. $24y^4 + 3y + 8y^3 + 1$ by $2y + 1$
17. $3x^3 - 5x^2 + 2$ by $x + 1$
18. $24m^4 + 16m^2 - 12m^3 + 17$ by $m + 1$
19. $x^2 - 3xy + 7y^2$ by $x - 7y$
20. $a^4 + 2a^3 + 3a^2 + 2a - 3$ by $a + 3$

CHAPTER 4

EQUATIONS OF FIRST DEGREE

The first step to solving a problem is to translate it into equations, and the second step is to solve those equations. In this chapter the methods of solving first-degree equations will be presented.

28. Kinds of Equations. There are two kinds of equations: equations of *identity* and equations of *condition*. An equation of identity expresses an equality that is true for all values of the letters it contains. For example, $6y \equiv 4y + 2y$ is an identity (an equation of identity) because it is true for any value of y, such as 2, 5, 3.7, or any value that may be given it. (The symbol $\equiv$ means "is identical to" and is used as the sign of an identity.)

An equation of condition expresses an equality that is true *only for certain values* of the letters it contains. For example, $x - 3 = 7$ is a conditional equation, since $x - 3 = 7$ *only* if $x = 10$. The equations used in solving problems are conditional equations. They are true only for certain values of the letters they contain, and *those values are the answers to the problem,* if the conditional equations correctly express the conditions of the problem. In this book the term *equation* will be used to mean a conditional equality, *i.e.*, an equation of condition.

29. Roots. To solve an equation that contains only one unknown, one finds the value of the unknown that reduces the equation to an identity. For example, the equation $x - 3a = 7a$ becomes $7a \equiv 7a$, an identity, if x is given the value $10a$.** The value $10a$ is said to *satisfy* the equation, and it is called a *solution* or *root* of the equation. The equation $x - 2 = 5$ reduces to $5 \equiv 5$, an identity, when $x = 7$; thus 7 is a root of this equation.

Equations are said to be *equivalent* when they have the same roots; *i.e.*, when they are satisfied by the same numerical values of the letters.

Example: $2x - 4 = 10$ and $x - 2 = 5$

are equivalent equations, since both reduce to identities if $x = 7$.

30. Operations on Equations. In the first course in algebra the following operations were used a great deal in the process of finding the solutions of equations:

1. Adding the same number (either arithmetic or general) to both members of an equation.
   (NOTE: The members, or "sides," of an equation are the expressions on each side of the sign of equality.)
2. Subtracting the same number (either arithmetic or general) from both members.
3. Multiplying both members by the same constant or arithmetic number (other than zero).
4. Dividing both members by the same constant or arithmetic number (other than zero).

These operations produce equations equivalent to the original equation (*i.e.*, satisfied by the same roots); hence they are permissible in solving equations.

Example 1. Solve: $x - 7 = 12$
Adding 7 to each member, $x + 0 = 19$
or $x = 19$

Example 2. Solve: $x + 7 = 12$
Subtracting 7, $x + 0 = 5$
or $x = 5$

Adding a number to both members of an equation (or subtracting it from both) is equivalent to *transposing* it, or moving it to the other side and *changing its sign*. For example, $x - 7 = 12$ becomes $x = 12 + 7$ when -7 is transposed (becoming $+7$ on the other side). Transposition is quite convenient in solving equations.

Example 3. Solve: $6x - 12 = 2x - 8$
Transposing -12, $6x = 2x - 8 + 12$
$6x = 2x + 4$
Transposing $2x$, $6x - 2x = 4$
$4x = 4$
Dividing both members by 4, $x = 1$

The solution can be checked by substituting the root 1 for x in the original equation:

$$\begin{aligned} 6x - 12 &= 2x - 8, \\ 6 - 12 &= 2 - 8, \\ -6 &\equiv -6, \text{ checks} \end{aligned}$$

Hereafter, the symbols T, M, D will be used at the left of the equation to indicate that the accompanying number is to be transposed, multiplied into both members, or divided into both members.

Example 4. Solve for x: $4x - 3a = x + 6a$

$$\begin{array}{lrl} \mathrm{T}(-3a), & 4x &= x + 6a + 3a \\ & 4x &= x + 9a \\ \mathrm{T}(x), & 4x - x &= 9a \\ & 3x &= 9a \\ \mathrm{D}(3), & x &= 3a \end{array}$$

Here, $3a$ is the *formular* value of x in terms of a. Given the desired value of a, one can compute the value of x directly. Of course, $3a$ is a solution, or root, of the original equation.

Example 5. Solve for x: $5x - 2a + 7 = 8a + 27$

$$\begin{array}{lrl} \mathrm{T}(-2a) \text{ and } (7), & 5x &= 8a + 27 + 2a - 7 \\ & 5x &= 10a + 20 \\ \mathrm{D}(5), & x &= 2a + 4 \end{array}$$

$2a + 4$ is a root of the original equation, and $x = 2a + 4$ is the formular value of x in terms of a.

The steps used in solving a simple equation containing one unknown can be summarized as follows:

1. Remove parentheses, if there are any in the equation.

2. Transpose all terms containing the unknown to one side of the equation, and transpose all other terms to the other side.

3. Collect similar terms on each side of the equation.

4. Divide both members of the equation by the coefficient of the unknown.

EXERCISES (30)

Solve the following equations:

1. $3x + 7 = x + 19$
2. $2x = 7x - 15$
3. $y - 6 = 5y + 14$
4. $2x - 4 = 6x - 12$
5. $6 - 9x = 3 - 8x$
6. $4(x - 1) = 3(x + 1)$

7. $8a + 9 = 3a - 6$
8. $2(x - 1) = x + 1$
9. $9x + x = 2 + 2x$
10. $-7 - 3x = 15 + 2x$
11. $5(x - 2) = 6(x - 3)$
12. $6c + 9 = c + 6$
13. $20y = 50(5 - y)$
14. $30m + 10(16 - m) = 200$
15. $3(x + 1) + x^2 = x^2 + 12$
16. $(x - 3)(x + 1) = (x - 2)(x - 1)$
17. $32(6 - 2x) = 15x$
18. $(2x - 1)(x - 7) - 2(x^2 - 3) = 25$

Solve for x:

19. $7x - 5a = 2x + 10a$
20. $3x + 7a + 6 = x - 5a + 8$
21. $2(3x - 5a) + 7 = 4x + 12a - 17$
22. $(x + 3)(x - 2)(x - 4) = x^2(x - 3) + 4$
23. $6a^2 + 12x - 2a + 25 = 10x + 10a + 15$
24. $(2x + 3)^2 - x(4x - 1) = 5x - 13 + 2(1 - x)$

31. Equations in Two Unknowns.

Example 1. Solve for x in the equation

$$2x - 7 = x + y + 8$$

Solution: $T\ (-7)$, $\quad 2x = x + y + 15$

$T\ (x)$, $\quad 2x - x = y + 15$

$$x = y + 15$$

The solution $y + 15$ represents the value of x that satisfies the original equation. There are as many satisfactory values for x as there are choices of values for y, *viz.*,

if $\quad y = 5, \quad x = 5 + 15 = 20$

if $\quad y = 2, \quad x = 2 + 15 = 17$, etc.

The equation $x = y + 15$ does not yield a specific value for x, unless a specific value for y is given. This is true of any equation that contains two unknown or unspecified quantities. A single equation in two unknowns will yield a formula for one in terms of the other, but *not* a numerical answer.

32. Pairs of Equations. Consider the following problem:

Example 1. Find two numbers whose sum is 30, and whose difference is 6.

Solution: Let x and y be the two numbers. Since their sum is 30, we may write

$$x + y = 30 \qquad \text{Equation (1)}$$

Since the difference of the two numbers is 6, we may write

$$x - y = 6 \qquad \text{Equation (2)}$$

The conditions of the problem are now represented by the two equations.

$$x + y = 30 \qquad (1)$$
$$x - y = 6 \qquad (2)$$

There are many pairs of numbers (x and y) whose sum is 30; *e.g.*, $x = 10$ and $y = 20$, or $x = 5$ and $y = 25$, or $x = 22$ and $y = 8$, etc. (Check these values by substituting them in the first equation.) Also, there are many pairs of numbers whose difference ($x - y$) is 6; such as $x = 10$ and $y = 4$, or $x = 16$ and $y = 10$, etc. Of the many pairs of numbers whose sum is 30, *there is only one pair whose difference is* 6; *i.e.*, there is only one pair of numbers whose sum is 30 *and* whose difference is 6. That pair of numbers, obviously, is the pair we wish to find; and it is also the pair that satisfies both of the equations (1) and (2).

Now let us solve (2) for x, as follows:

$$x - y = 6 \qquad (2)$$
$$\text{T}(-y), \qquad x = 6 + y \qquad (3)$$

Note that (2) has been solved for x *in terms of* y, but not for a numerical value of x. In (1), we can now replace x by the expression $6 + y$:

$$(6 + y) + y = 30$$
$$\text{or} \qquad 6 + 2y = 30$$
$$\text{T}(6), \qquad 2y = 24$$
$$\text{D}(2), \qquad y = 12$$

This is the desired numerical value of y. We now replace y by this value in any of the equations (1), (2), or (3). Using (1),

$$x + 12 = 30$$
$$\text{T}(12), \qquad x = 30 - 12$$
$$x = 18$$

Observe that the numerical values $x = 18$ and $y = 12$ satisfy both of the original equations:

$$18 + 12 \equiv 30 \qquad \text{and} \qquad 18 - 12 \equiv 6$$

which are the conditions of the original problem: "Find two numbers *whose sum is* 30 and *whose difference is* 6."

33. Elimination by Substitution. The method used in the preceding example is called *elimination by substitution.* Observe that

the expression $6 + y$ was *substituted* for x in (1) and that the result was an equation that contains only y. In other words, the substitution of $6 + y$ for x served to *eliminate* x from the equation and made it possible to obtain a numerical value for y.

> ***Elimination by Substitution***
>
> ***In one equation, obtain the value of one unknown in terms of the other; then substitute this value into the other equation.***

Example 1. Solve:

$$3x - y = 5 \qquad (1)$$
$$x + 4y = 6 \qquad (2)$$

Solution: Transposing $4y$ in (2),

$$x = 6 - 4y$$

Substituting $6 - 4y$ for x in (1),

$$3(6 - 4y) - y = 5$$
$$18 - 12y - y = 5 \quad \text{or} \quad 18 - 13y = 5$$
$$\text{T}(18), \quad -13y = 5 - 18$$
$$-13y = -13$$
$$\text{D}(-13), \quad y = 1$$

Substituting $y = 1$ in (1),

$$3x - 1 = 5$$
$$\text{T}(-1), \quad 3x = 6$$
$$\text{D}(3), \quad x = 2$$

Checking,
$$3x - y = 5, \quad 6 - 1 \equiv 5$$
$$x + 4y = 6, \quad 2 + 4 \equiv 6$$

Solving for the values of the unknowns that satisfy both of two equations, or all of a group of equations, is called *solving the system* (of equations), or solving the equations *simultaneously*.

Example 2. Solve the system:

$$3x - 2y = 2 \qquad (1)$$
$$\underline{6x - 5y = -1} \qquad (2)$$

Solving (1) for x,

$$3x = 2y + 2$$
$$\text{D}(3), \quad x = \frac{2y + 2}{3}$$

Substituting in (2),

$$6\left(\frac{2y+2}{3}\right) - 5y = -1$$

$$2(2y+2) - 5y = -1$$

$$4y + 4 - 5y = -1$$

T(4), $$4y - 5y = -1 - 4$$

$$-y = -5$$

D(−1), $$y = 5$$

Substituting $y = 5$ in (1),

$$3x - 10 = 2$$

T(−10), $$3x = 12$$

D(3), $$x = 4$$

Checking, $3x - 2y = 2$ becomes $12 - 10 \equiv 2$
$6x - 5y = -1$ becomes $24 - 25 \equiv -1$

EXERCISES (33)

Solve the following pairs of equations, eliminating by substitution. Check your answers to the even problems by substituting them in *both* the original equations. If in error, they may satisfy one of the equations but not both.

1. $x - 2y = 6$ $4x + y = 6$	2. $2x - y = 2$ $x - 2y = -11$	3. $x + 2y = 14$ $2x + y - 13 = 0$
4. $x + y = 5$ $5x - 2y = 4$	5. $x - 3y - 13 = 0$ $3x + y - 7 = 0$	6. $2x - y = 7$ $x + y = 8$
7. $7m - 2n = 34$ $8m - 3n = 8$	8. $3x - 4y = 8$ $2x + 2y = 10$	9. $3x - 4y = 16$ $2x + y = 28$
10. $m - 3n = 17$ $2m - n = 4$	11. $7x - 4y = 20$ $6x - 2y = 10$	12. $23x - 17y = 58$ $7x - 14y = 12$

34. Elimination by Addition or Subtraction. In solving pairs of equations, it is often possible to eliminate one unknown by simply adding or subtracting the corresponding members of the two equations:

Example 1. Solve the system: $2x - y = 11$ (1)

$$3x + y = 4 \qquad (2)$$

Solution: Adding, $$5x + 0 = 15$$

D(5), $$x = 3$$

Substituting 3 for x in (1),

$$6 - y = 11$$

T(6), $$-y = 5$$

D(−1), $$y = -5$$

Thus $x = 3$ and $y = -5$. Checking by substitution in the original equations,

$$6 - (-5) \equiv 11$$
$$9 - 5 \equiv 4$$

Example 2. Solve:

$$5x + 2y = 46 \qquad (1)$$
$$3x + 2y = 38 \qquad (2)$$

Solution: Subtracting, $2x + 0 = 8$

D(2), $x = 4$

The value of y can be found by substituting 4 for x in either of the original equations. Complete the solution.**

In most cases it is not possible to eliminate one unknown by direct addition or subtraction.

Example 3. Solve:

$$5x - 3y = 83 \qquad (1)$$
$$x + y = 7 \qquad (2)$$

Since the coefficients of x in these equations are unequal in size, as is true also of the coefficients of y, addition or subtraction will result in an equation that contains both x and y. In this example, addition produces the equation $6x - 2y = 90$, which is of no more value to us than one of the original equations. Suppose, however, that (2) is multiplied (on each side) by 3, giving

$$3x + 3y = 21 \qquad (3)$$

Now this equation can be added to (1):

$$5x - 3y = 83 \qquad (1)$$
$$3x + 3y = 21 \qquad (3)$$
$$8x = 104 \qquad (4)$$

D(8), $x = 13$

Substituting $x = 13$ in (2),

$$13 + y = 7$$
$$y = -6$$

Observe that it is necessary to obtain two equations in which the coefficients of the unknown to be eliminated are of equal size, if it is to be eliminated by addition or subtraction.

In the preceding example, we could have eliminated x instead of y by multiplying (2) by 5 and subtracting from (1):

$$5x - 3y = 83 \qquad (1)$$
$$5x + 5y = 35 \qquad (2), \text{ multiplied by } 5$$

Subtracting, $\quad -8y = 48$

D(−8), $\quad y = -6$

Example 4. Solve:

$$4x - 3y = 17 \qquad (1)$$
$$3x - 2y = 42 \qquad (2)$$

Here it is necessary to multiply the first equation by 2 and the second by 3 in order to eliminate y, or to multiply the first by 3 and the second by 4 to eliminate x.

$4x - 3y = 17$ multiplied by 2 is $8x - 6y = 34$

$3x - 2y = 42$ multiplied by 3 is $9x - 6y = 126$

Subtracting, $\quad -x = -92$

$$x = 92$$

Substituting 92 for x in (1),

$$368 - 3y = 17$$

T(368), $\quad -3y = -351$

D(−3), $\quad y = 117$

Checking by substitution in the original equations,

$$368 - 351 \equiv 17 \quad \text{and} \quad 276 - 234 \equiv 42$$

> ***To eliminate one unknown by addition or subtraction, multiply each equation (if it is necessary to do so) by such positive numbers as will make the coefficients of that unknown have the same absolute value in the two equations. Then add the resulting equations or subtract them, whichever is necessary in order to eliminate the unknown. If one of the unknowns has the coefficient 1 in one or both equations, that unknown is usually the one to eliminate, since only one multiplication (or none) is required.***

EXERCISES (34)

Solve the following pairs of equations, eliminating by addition or subtraction:

1. $2m - 5n = 4$ $3m + 5n = 11$	2. $2x - y = 7$ $x + y = 8$	3. $2x + 3y = 7$ $3x + y = -7$
4. $5x + 2y = 12$ $7x + 4y = 12$	5. $x - 3y - 13 = 0$ $3x + y - 7 = 0$	6. $3x - 5y = 13$ $x + y = -11$
7. $7x - 2y = 34$ $8x - 3y = 8$	8. $3x - 4y = 7$ $2x + 2y = 8$	9. $3u - 4v = 16$ $2u + v = 28$

10. $x - 3y = 17$
$2x - 2y = 8$

11. $3x - 5y = 13$
$6x - 7y = -8$

12. $7x + 4y - 12 = 0$
$6x + 2y - 42 = 0$

Solve the following pairs of equations by any method you choose. Check your answers.

13. $x + 2y = 5$
$4x + y = 6$

14. $4m + 7n = 23$
$m - 7n = 21$

15. $6x + 5y = 21$
$7x - 5y = 5$

16. $4a - 2b = 4$
$a + 5b = 23$

17. $2x + 3y = 9$
$3x - 2y = 7$

18. $7a - b = 9$
$2a - b = 5$

19. $7x + 3y = 54$
$4x - 2y = 12$

20. $8x + 6y = 10$
$5x + 2y = 1$

21. $7x - 4y = 12$
$8x - 5y = 0$

22. $4a - 3b = 90$
$a - 2b = 45$

23. $2x + 3y = 15a$
$x - y = 3a$

24. $3x - y = 2c$
$4x + 2y = 16c$

NOTE: The solution of problems in which x and y are to be determined in terms of other general numbers, say m and n, is said to yield *formular* values of x and y, or formulas for computing the numerical values of x and y for any given values of m and n. Determine formular values for x and y in the following:

25. $2x + 3y = a$
$3x - 2y = b$

26. $ax - 2y = 12$
$bx + 3y = 20$

27. $mx + ny = a$
$mx - ny = b$

28. $ax + by = e$
$cx + dy = f$

35. Equations Involving Three Unknowns. Consider the following problems:

Example 1. A man has a number of coins in his pocket, worth \$1.56 in all. They include pennies, nickels, and dimes. The number of nickels is two more than twice the number of dimes, and the number of pennies is two less than twice the number of nickels. How many of each has he?

Solution: Let x be the number of pennies, y the number of nickels, and z the number of dimes. Their total value is $x + 5y + 10z = 156$. Also, the number of nickels is two more than twice the number of dimes, or two more than $2z$; so $y = 2z + 2$. The number of pennies is two less than twice the number of nickels, or two less than $2y$; so that $x = 2y - 2$. We now have three equations:

$$x + 5y + 10z = 156 \quad (1)$$
$$y = 2z + 2 \quad (2)$$
$$x = 2y - 2 \quad (3)$$

The first equation contains all three unknowns; the others contain two

each, but not the same two. In order to obtain a numerical answer for one of the unknowns, we must obtain an equation that contains only one unknown. This may be done by determining the values of x and z in terms of y, by means of the second and third equations, and replacing x and z in the first equation by these values. It will then contain only y, which can be found. This procedure constitutes elimination by substitution, which has already been used for cases involving only two unknowns.

Solving (2) for z,

$$2z + 2 = y \qquad (2), \text{ reversed}$$

$$\text{T}(2), \qquad 2z = y - 2$$

$$\text{D}(2), \qquad z = \frac{y-2}{2} \qquad (4)$$

Equation (3) gives x in terms of y directly: $x = 2y - 2$. We shall now replace z by $\frac{y-2}{2}$ and x by $2y - 2$ in (1):

$$x + 5y + 10z = 156 \qquad (1)$$

$$(2y - 2) + 5y + 10\frac{(y-2)}{2} = 156$$

$$2y - 2 + 5y + 5(y - 2) = 156$$

$$2y - 2 + 5y + 5y - 10 = 156$$

$$12y - 12 = 156$$

$$\text{T}(-12), \qquad 12y = 168$$

$$\text{D}(12), \qquad y = 14$$

Putting $y = 14$ in (3),

$$x = 28 - 2 = 26$$

Putting $y = 14$ in (4),

$$z = \frac{y-2}{2} = \frac{14-2}{2} = \frac{12}{2} = 6$$

Thus there are 26 pennies, 14 nickels, and 6 dimes, worth

$$26 + 70 + 60 = 156 \text{ cents} = \$1.56$$

Example 2. You have 29 coins (pennies, nickels, and dimes) worth \$1.27. The number of nickels and dimes is 5 more than the number of pennies. How many of each have you?

Solution: Let x = pennies, y = nickels, and z = dimes as before.

Then

$$x + 5y + 10z = 127 \qquad (1)$$

$$x + y + z = 29 \qquad (2)$$

$$y + z = x + 5 \qquad (3)$$

Elimination by substitution is not usually convenient when all three equations contain all three unknowns, as in this case. Let us use addition and subtraction to eliminate some of the unknowns. First, let us rearrange the equations with similar terms in the same order. In this problem, only (3) needs to be rearranged, and after transposing x, it is

$$-x + y + z = 5$$

Now the three equations are

$$x + 5y + 10z = 127 \qquad (1)$$
$$x + y + z = 29 \qquad (2)$$
$$-x + y + z = 5 \qquad (3)\text{, rearranged}$$

In order to eliminate two of the unknowns by addition and subtraction, we must first eliminate the same unknown *twice* by combining two of the equations at a time. Thus, let us add (2) and (3) to eliminate x, obtaining $2y + 2z = 34$, and also subtract (2) from (1) to eliminate x, obtaining $4y + 9z = 98$. From here on the problem is simple, since we have *two* equations in *two* unknowns:

$$2y + 2z = 34 \qquad (4)$$
$$4y + 9z = 98 \qquad (5)$$

Multiplying (4) by 2 and subtracting the resulting equation from (5),

$$4y + 9z = 98 \qquad (5)$$
$$\underline{4y + 4z = 68} \qquad (4)\text{, multiplied by 2}$$

Subtracting,
$$5z = 30$$
$$z = 6 \text{ dimes}$$

Substituting 6 for z in (4) (note that this is one of the two secondary equations that contain only two unknowns),

$$2y + 12 = 34$$
$$y = 11 \text{ (nickels)}$$

To obtain x we return to the original equations, selecting a simple one in which to substitute 6 for z and 11 for y to obtain the value of x. Substituting these values in (2),

$$x + 11 + 6 = 29$$
$$x = 12$$

Thus there are 12 pennies, 11 nickels, and 6 dimes.

Checking,
$$12 + 55 + 60 \equiv 127 \qquad (1)$$
$$12 + 11 + 6 \equiv 29 \qquad (2)$$
$$11 + 6 \equiv 12 + 5 \qquad (3)$$

As in the case of equations with two unknowns, it may be necessary to multiply one or more of the equations in three unknowns by certain positive numbers in order to cause one unknown to be eliminated when one adds or subtracts the equations.

EXERCISES (35)

Solve the following systems of equations in three unknowns. Eliminate by substitution or by addition and subtraction, choosing your method to suit the problem.

1. $x + 2y + 3z = 6$
   $x - 2y + 3z = 2$
   $x + 2y - 3z = 0$

2. $a + b = 10$
   $a + c = 12$
   $b + c = 16$

3. $x + 2y = 30$
   $2x + z = -28$
   $3y - 2z = 76$

4. $u + v - w = 2$
   $u + v + w = 4$
   $u - v + w = 6$

5. $4x + 3y + 8z = 5$
   $2x + 6y - 4z = 2$
   $6x + 3y - 4z = 3$

6. $a + c = 4$
   $a + b = 6$
   $b + c = 8$

7. $2x + 3y - z = 5$
   $3x - y + 2z = 7$
   $x + y + z = 6$

8. $3x - 4y = 6$
   $3x + 2y + 2z = 6$
   $x - 5y + 6z = 2$

9. $ax + by = 1$
   $cz + ax = 1$
   $by + cz = 1$

10. $3x + 2y - z = 4$
    $5x - 3y + 2z = 5$
    $6x - 4y + 3z = 7$

REVIEW QUESTIONS

1. What is an equation of condition?
2. What is a root of an equation?
3. What are equivalent equations?
4. What operations on equations are permissible in solving problems?
5. Why are they permissible?
6. Can x be 5 in the equation $2x - 10 = 7$?
7. Can x be 5 in the equation $x + y = 7$? Can it be 6? 7?
8. What are the possible values of x and y in the equation $x - y = 3$?
9. Can the equation $x - y = 3$ be solved? Explain.
10. If a stated problem yields two equations containing x and y, what must be done to obtain the numerical values of x and y?
11. How many independent equations are necessary in finding three unknowns?
12. How does one solve three equations containing three unknowns?

CHAPTER 5

STATED PROBLEMS

The student should solve a large number of the problems in this chapter, for the technique used in solving them is a most important device in all branches of engineering, science, and mathematics. It is also the foundation of all the mathematics to follow, both in this course and those that are more advanced. Sufficient time should be devoted to the material in this chapter to ensure thorough familiarity with the techniques employed in applying the algebraic method to the various types of problems listed.

36. General Method of Solving Stated Problems. The algebraic method of solving stated problems consists, in general, of two steps:

1. Choosing symbols to represent the quantities to be determined, then translating the conditions of the problem into as many independent equations as there are unknowns.
2. Solving the equations obtained in (1) for the values (either numerical or formular) of the quantities to be determined.

Use this method to solve the following problems.

PROBLEMS (36)

1. Find two numbers whose sum is 60 and whose difference is 14.

2. The sum of two numbers is 24, and one of the numbers is twice the other. What are the numbers?

3. A man has \$3.95 in nickels and dimes. How many of each has he if there are 55 coins in all?

4. The gate receipts at a football game were \$4,237.50 from 4,075 paid admissions. If reserved-seat tickets were \$1.50 each and general-admission tickets were \$.90, how many tickets of each kind were sold?

5. An airplane whose air speed is 225 m.p.h. requires 5 hr. to fly 1,000 miles against a head wind. What is the speed of the wind?

6. Two men have a total of \$340. One of the men has \$82 more than the other. How much does each have?

7. An automobile traveled 1,500 miles in 2 days. It traveled 135 miles farther on the second day than it did on the first. How far did it travel each day?

8. A grocer mixes two grades of coffee that he has been selling for

60 cents per pound and 40 cents per pound, respectively. How much of each must he take to make a mixture of 25 lb. to sell at 44 cents per pound? HINT: One of the equations should express the total value of the 25 lb. of coffee, just as in the stamp problems.

9. A man invested \$10,000, part at 2% interest and part at 3%. The investments yielded \$275 in a year. How much was invested at each rate?

10. A man made two investments, totaling \$12,000. He profited 10% on the first, but lost 5% on the second. His net profit was \$75. What was the amount of each investment?

11. A man invests \$2,000 more at 4% than he does at 3%. The total interest on the two investments amounts to \$353 per year. How much is invested at each rate?

12. Two cars were 200 miles apart after traveling in opposite directions for 4 hr., starting at the same point. If one traveled 5 m.p.h. slower than the other, find the speed of each.

13. Two hours after an airplane left an airport another plane started in pursuit of it. Six hours were required for the second plane to catch the first. If the second plane traveled 45 m.p.h. faster than the first, what were their speeds?

14. A man started to a town $52\frac{1}{2}$ miles away, walking 3 m.p.h. After walking part way, he was picked up by a motorist who took him to his destination at 30 m.p.h. If the entire trip required 4 hr., how far did he walk and how far did he ride? HINT: One of your equations must be a time equation.

15. One automobile went 20 miles farther, traveling 50 m.p.h., than a second one that traveled 3 hr. longer at 40 m.p.h. How many hours did each travel?

16. A man has 50 cents more than enough to buy 8 Meadowlark golf balls. He would need 70 cents more in order to buy 10 Longflite golf balls which are 5 cents higher in price than the Meadowlark balls. Determine the prices of the two kinds of balls.

17. Find two numbers N_1 and N_2 whose sum is S and whose difference is D.

18. Find the speeds S_1 and S_2 of two automobiles that start from the same point in opposite directions and are a distance D miles apart in a time T hr., if one automobile travels twice as fast as the other. NOTE: For each automobile, the distance traveled is the speed times the time.

19. Find the speeds S_1 and S_2 of two automobiles that start from the same point and are a distance D apart in a time T hr., if one automobile travels M m.p.h. faster than the other.

20. Find three numbers such that the sum of the first and second is

52, the sum of the second and third is 96, and the sum of the first and third is 100.

21. The sum of the three angles of any triangle is 180°. In a certain triangle, angle A is 15° greater than angle B, and angle B is 27° greater than angle C. Find the angles.

37. A Timesaving Technique. Thus far the student has been encouraged to solve stated problems in the most straightforward way, *i.e.*, by assigning a symbol to each unknown and obtaining as many equations as unknowns. When the relationship between the unknowns is very simple, one can do the equivalent of solving one equation mentally and eliminating one unknown by substitution. Consider the problem:

Example 1. Find two numbers whose difference is 3 and whose sum is 27.

Solution: Let the numbers be x and y.

Then $\qquad x - y = 3 \qquad (1)$

and $\qquad x + y = 27 \qquad (2)$

From (1), $\qquad x = y + 3$

and we can substitute $y + 3$ for x in (2), obtaining

$$(y + 3) + y = 27$$

which can easily be solved for y.

Now for the timesaving method: In reading the problem, we understand the phrase "whose difference is 3" to mean that one of the numbers is 3 more than the other. Thus, we can let y (or any letter) be one of the numbers and let $y + 3$ be the other one, writing $y + (y + 3) = 27$ *at once.* Completing the solution,

$$2y + 3 = 27$$
$$y = 12$$

and the numbers are 12 and 15.

The steps involved in this shorter method of solving a stated problem in two unknowns are as follows:

1. Represent one of the unknowns by a symbol, such as x.
2. Observe the simpler of the two relations between the unknowns and use it to represent the other unknown in terms of the same symbol.

3. Write the equation that expresses the remaining condition of the problem.
4. Solve the equation for the first unknown; then evaluate the second.

This technique of representing several unknowns in terms of the same symbol is particularly useful in connection with certain more difficult types of problems, such as those involving three or more simply related unknowns:

Example 2. Find three *consecutive* numbers whose sum is 105.

Solution: In this problem, the relationship among the three unknowns is very simple. If x is the smallest, the next larger is equal to $x + 1$, and the third is equal to $x + 2$. The sum of the three numbers is

$$\begin{aligned} x + (x + 1) + (x + 2) &= 105 \\ 3x + 3 &= 105 \\ x &= 34 \end{aligned}$$

PROBLEMS (37)

Solve the following by representing all the unknowns in terms of a single symbol, thus obtaining and solving only one equation:

1. Find two numbers whose sum is 38 and whose difference is 11.

2. The sum of two numbers is 96, and one of the numbers is twice the other. What are they?

3. A man has \$6.55 in quarters and dimes. How many of each has he if there are 40 coins in all?

4. The gate receipts at a football game were \$7,200, from 5,000 paid admissions. If reserved-seat tickets were \$1.75 each and general admission tickets were \$1.25, how many tickets of each kind were sold?

5. Two men have a total of \$570. One of the men has \$187 more than the other. How much does each have?

6. An automobile traveled 1,450 miles in 2 days. It traveled 126 miles farther on the second day than it did on the first. How far did it travel each day?

7. A grocer mixes two grades of coffee that he has been selling for 50 cents per pound and 40 cents per pound, respectively. How much of each must he take to make a mixture of 25 lb. to sell at 44 cents per pound?

8. A man invested \$20,000, part at 2% interest and part at 3%. The investments yielded \$440 in a year. How much was invested at each rate?

9. A man made two investments totaling \$16,000. He profited 10% on the first, but lost 5% on the second. His net profit was \$100. What was the amount of each investment?

10. A man invests \$2,000 more at 4% than he does at 3%. The total interest on the two investments amounts to \$388 per year. How much is invested at each rate?

11. Two cars were 500 miles apart after traveling in opposite directions for 5 hr., starting at the same point. If one traveled 4 m.p.h. slower than the other, find the speed of each.

12. Two hours after an airplane left an airport another plane started in pursuit of it. Eight hours were required for the second plane to catch the first. If the second plane traveled 55 m.p.h. faster than the first, what were their speeds?

13. A man started to a town $55\frac{1}{2}$ miles away, walking 3 m.p.h. After walking part way, he was picked up by a motorist who took him to his destination at 30 m.p.h. If the entire trip required 5 hr., how far did he walk and how far did he ride?

14. Find three consecutive numbers whose sum is 45.

15. Find three consecutive *even* numbers whose sum is 150.

16. The second of three numbers is two more than twice the first, the third is three less than three times the first, and their sum is 173. Find the three numbers.

38. Classification of Stated Problems. Most of the problems thus far presented can be solved by using either one or two symbols in representing the unknown quantities:

Example 1. A purse contains \$2.45 in nickels and dimes, of which there are 35 in all. How many of each are there?

Two Symbols

x = number of nickels
y = number of dimes

$$x + y = 35 \qquad (1)$$
$$5x + 10y = 245 \qquad (2)$$

Multiplying (1) by 5 and subtracting from (2),

$$5x + 10y = 245 \qquad (2)$$
$$5x + 5y = 175 \qquad (3)$$
$$5y = 70$$
$$y = 14 \text{ (dimes)}$$

Substituting in (1),

$$x + 14 = 35$$
$$x = 21 \text{ (nickels)}$$

One Symbol

x = number of nickels.
There are 35 coins in all; hence the number of dimes is $35 - x$. Then

$$5x + 10(35 - x) = 245$$
$$5x + 350 - 10x = 245$$
$$-5x = -105$$
$$x = 21 \text{ (nickels)}$$
$$35 - x = 35 - 21 = 14 \text{ (dimes)}$$

Some problems are not easy to solve by representation in terms of one unknown.

Example 2. If 30 lb. of tea and 20 lb. of coffee cost \$24.60, and 10 lb. of tea and 25 lb. of coffee cost \$28.25, what are the prices of tea and coffee?

Solution: Let x represent the price of tea. It would be difficult to represent the price of coffee in terms of x without writing and solving an equation; hence it is best to let y represent the price of coffee and write the two equations

$$30x + 20y = 2{,}460 \tag{1}$$
$$10x + 25y = 2{,}825 \tag{2}$$

and the solution can be completed without difficulty.

The most direct method of solving stated problems in which two quantities are to be determined is to represent one of the unknown quantities by a symbol, then see if it is convenient to represent the other unknown in terms of the same symbol. If so, the problem can be solved by writing a single equation. If not, a separate symbol must be assigned to each unknown and two equations must be written and solved.

There are four types of problems that must not be confused.

1. Problems in one unknown.

Example: An airplane whose air speed is 225 m.p.h. flies 1,000 miles in 5 hr. against a head wind. What is the speed of the wind?
In this problem only one quantity, the speed of the wind, is unknown. One therefore writes $(225 - x)(5) = 1{,}000$ and solves for x.

2. Problems in two unknowns, one of which can conveniently be represented in terms of the other, thus leading to a single equation in one unknown.

Example: Find two numbers whose difference is 6 and whose sum is 30.
This type of problem has been discussed.

3. Problems in two unknowns that are not easy to solve by the method of (2). These should be solved by means of two equations in two unknowns, with the unknowns being represented by separate symbols. This type has also been illustrated.
4. Problems in two unknowns, only one of which is to be determined. This type of problem causes no difficulty when the extra unknown is readily expressed in terms of the one to be found. The method

of (2) is used to obtain a single equation involving only the unknown whose value is desired. One is not likely to realize that a problem of this type involves two unknowns.

Example: A man invests \$2,000 at 2%. How much should he invest at 5% in order to make 3% on his total investment?

Here there are two unknowns, the amount x to be invested at 5% and the total amount invested; but the total investment is so easy to express as $x + \$2{,}000$ that the process is automatic. One writes

$$(.02)(\$2{,}000) + (.05)(x) = .03(x + 2{,}000)$$

and solves for x.

When the extra unknown is not easily represented in terms of the one that is to be found, the problem may be very troublesome.

Example: An airplane flies 720 miles in 4 hr. *with* the wind, then returns in 5 hrs. What is the speed of the wind?

Solution: There are two unknowns, the speed of the wind and the air speed of the plane; but the latter cannot easily be represented in terms of the former. It is necessary to use two symbols, S_w for the speed of the wind and S_p for the air speed of the plane, and to write two equations:

$$720 = 4(S_p + S_w)$$
$$720 = 5(S_p - S_w)$$

from which S_w can be determined by eliminating S_p.

39. Hidden Unknowns. Whenever you find it difficult to analyze a problem in which only one quantity is to be determined, check to see if you are overlooking a second unknown (a *hidden* unknown) that must be represented. If it can be represented in terms of the first unknown, well and good; if not, assign it a symbol and write two equations.

If two symbols are used when only one of the unknowns is to be determined, it is best (when possible) to write equations expressing the value of the unknown that is *not* to be determined, since it can then be eliminated without difficulty.

Example 1. How much cream which is 25% butterfat should be added to 50 gal. of low-grade milk (2.5% butterfat) to produce standard milk at 4% butterfat?

Solution: Let x be the amount of cream to be added. The *hidden unknown* is the total amount of butterfat (in the final mixture), which can be represented by the symbol B. Now let us write equations for B,

since we want to eliminate it. The total amount of butterfat in the final mixture is

$$B = .04(50 + x) \tag{1}$$

since there are (in all) $50 + x$ gal. at 4% butterfat. But B also equals the butterfat in the cream plus that in the low-grade milk, or

$$B = .25x + (.025)(50) \tag{2}$$

i.e., x gal. of cream (25% butterfat) contain $.25x$ gal. of butterfat, and 50 gal. of low-grade milk (at 2.5%) contain $(.025)(50)$. Since the left members of (1) and (2) are identical, we can equate their right members:

$$\begin{aligned} .04(50 + x) &= .25x + (.025)(50) \\ 2.0 + .04x &= .25x + 1.25 \\ .75 &= .21x \\ x &= 3.57 \text{ (gallons of cream)} \end{aligned}$$

If one recognizes the hidden unknown, he need not always use a symbol for it. In the preceding example, one could reason as follows: The total butterfat equals $.04(50 + x)$; *also* it equals $.25x + (.025)(50)$.

Thus, $$.04(50 + x) = .25x + (.025)(50)$$

Problems involving mixtures often involve a hidden unknown, as do many motion problems. In the latter, the amount of time required is often the hidden unknown.

40. Problems. Solve the following problems by any of the methods at your disposal. If you find it very difficult to obtain the proper equation or equations, refer to the type problems that are worked out as examples at the end of the chapter. Do *not* refer to them, however, unless you need to do so, for it is much better to arrive at the proper technique for a given problem through your own efforts. In practical applications, one does not always have a type problem at hand.

A. Value Problems

1. Six horses and eight mules cost \$2,712, while eight horses and six mules cost \$2,776. Assuming a single price for horses and a single price for mules, what are the prices?

2. You have a number of coins, some dimes and some quarters, worth \$6.25. If the dimes were quarters and the quarters were dimes, you would have \$1.50 more. How many of each do you have?

3. The receipts at a football game were \$5,040, for 4,800 paid admissions. If reserved-seat tickets were \$1.50 and general admission tickets were \$.90, how many of each were sold?

4. A boy has 10 more nickels than dimes, and he has twice as many nickels as pennies. If he has \$3.03 in all, how many of each does he have?

5. A dealer bought 50 dozen oranges at 22 cents per dozen. He sold them at 38 cents per dozen, but lost some through spoilage. If his profit was \$6.86, how many oranges spoiled?

6. One woman buys 30 yd. of silk and 38 yd. of linen for \$37.24. Another buys 15 yd. of silk and 24 yd. of linen for \$19.77. Find the prices of the two materials.

7. A man has 50 cents more than he needs to purchase 12 Meadowlark golf balls. He would need \$1 more in order to purchase 10 Longflite balls, which cost 25 cents more (each) than Meadowlark balls. What are their prices and how much money has he?

8. At an ice-cream bar, 1,280 ice-cream cones were sold for \$74.40. How many were 5-cent cones and how many were 10-cent cones?

9. At the post office a man purchased stamps worth \$4.50. He bought three times as many 3-cent stamps as 5-cent stamps, and 15 more 3-cent stamps than 8-cent stamps. How many of each did he buy?

10. A fruit dealer bought a number of apples at 20 cents a dozen. He sorted them into two groups, prime and second grade, and sold the prime apples at 45 cents a dozen, the second-grade apples at 35 cents a dozen. If his profit was \$13 and if there were twice as many prime apples as second grade, how many did he buy in the first place?

B. Age Problems

11. A father is 24 years older than his son, and in 5 years he will be exactly twice his son's age. What are their present ages?

12. A family has two children. The sum of their ages is 21. In 2 years one will be four times as old as the other. What are their ages?

13. Twelve years ago Mr. Jones was five times as old as Harry. Today he is twice as old as Harry. Find their ages.

14. Mr. Brown is twice as old as his son Ronald. Five years from now the sum of their ages will be five times what Ronald's age was 4 years ago. Find their ages.

15. A man 42 years old has two sons, one of whom is three times as old as the other. In 4 years, the sum of all their ages will be 86 years. How old are the boys?

16. A woman has two daughters, one 7 years older than the other. In two years the mother will be seven times as old as her younger

daughter, who will then be half as old as the older daughter. Find their ages.

C. Motion Problems

17. Two airplanes start toward each other from points 1,800 miles apart and meet 5 hr. later. If one flies 80 m.p.h. faster than the other, what are their speeds?

18. Two automobiles start in opposite directions from Oklahoma City. One travels 20 m.p.h. faster than the other, and at the end of 4 hr. they are 360 miles apart. Find their speeds.

19. A man drives 400 miles in 10 hr., going part of the way at 35 m.p.h. and the rest at 50 m.p.h. How far does he drive at each speed?

20. A man started toward a city 72 miles away, walking at 3 m.p.h. Later he was picked up by a motorist who took him to the city at 42 m.p.h. If the entire trip required $4\frac{1}{2}$ hr., how far did the man walk?

21. Two automobiles started in opposite directions from Cleveland and in 6 hr. are 480 miles apart. If the slower one had gone twice as fast as it did, they would have been 440 miles apart in 4 hr. How fast did the slower one travel?

22. Two airplanes start from Kansas City for Chicago, one traveling 10% faster than the other. At the end of 4 hr., they are 80 miles apart. Find their speeds.

23. A mail plane took off from St. Louis, traveling 180 m.p.h. Just after it rose from the ground, one of its landing wheels dropped off. The ground crew immediately prepared another plane, which took off 10 min. later to overtake and warn the pilot of the first plane. If the second plane flew at 240 m.p.h., how long did it take it to overtake the first? How far from St. Louis did it overtake the first plane?

24. An airplane whose air speed is 225 m.p.h. goes 875 miles in 5 hr. against a head wind. What is the speed of the wind?

25. A plane flies 960 miles in 3 hr. with the wind, then flies back in 4 hr. What are the values of the speed of the wind and the air speed of the plane?

26. A plane whose air speed is 250 m.p.h. has enough gasoline (besides that in the emergency tank) for 15 hr. of flight. How far can it fly with a 50-m.p.h. tail wind and still have enough gas to return without drawing on the emergency supply? Compare this cruising range with that when there is no wind.

27. If in Prob. 26 the wind should increase to 75 m.p.h. just as the plane starts to return, how many miles would the plane be from home when it started using the emergency supply of gasoline?

28. A plane flies 720 miles in 4 hr. with the wind, then flies back in 5 hr. What is the speed of the wind?

29. A plane can make a certain trip in 7.5 hr. against a 50-m.p.h. wind, or in $6\frac{2}{3}$ hr. against a 25-m.p.h. wind. What is the distance?

D. Mixture Problems

30. A druggist has on hand 2% and 5% solutions of tincture of iodine. A customer orders 3 oz. of 3% solution. How much of each should the druggist mix in order to fill the order?

31. A goldsmith wishes to make 200 grains of an alloy that is 75% gold. He has stock alloys that are 50% and 90% gold, respectively. How much of each should he use?

32. A druggist wishes to dilute a 5% iodine solution with pure alcohol to obtain 20 oz. of $3\frac{1}{2}$% solution. How much 5% solution and how much alcohol must he mix?

33. A dealer in perfumes wishes to blend perfume that sells for \$9 an ounce with 150 oz. of \$2 perfume, to make a blended perfume selling for \$2.50 an ounce. How much of the \$9 perfume must he use?

34. How much alcohol must be added to a mixture of 8 oz. of alcohol and 20 oz. of water to produce a mixture that is 75% alcohol?

35. A confectioner has 30 lb. of Brazil nuts selling for 50 cents per pound and 25 lb. of walnuts selling at 80 cents per pound. How many pounds of almonds at 70 cents a pound should he add to the walnuts and Brazil nuts to make a mixture selling at 65 cents a pound? at 75 cents a pound?

36. How much cream that is 28% butterfat must be added to 20 gal. of milk that is 3% butterfat to obtain grade A milk at 4% butterfat?

37. A chemist has two solutions of the same reagent in different strengths. Three parts of one mixed with 2 parts of the second produce a 60% solution, while 2 parts of the first and 3 parts of the second produce a 50% solution. What are the strengths (in per cent) of the two solutions?

38. A druggist has 200 cc. of 2% tincture of iodine solution and an unlimited supply of 5% solution. He wishes to mix enough 2% solution and 5% solution to produce a 4% solution, and he wants the amount of 4% solution to equal the remaining amount of 2% solution. How much 2% and how much 5% solution should he mix?

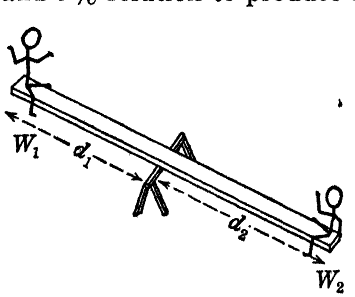


FIG. 3.—The teeter board.

E. Problems in Mechanics

39. Two boys are balanced on a teeter board as shown in Fig. 3. It is a well-known principle of physics that whenever two objects balance on

a lever, the products obtained by multiplying the weight of each by its distance from the point of support are equal; *i.e.*, $W_1d_1 = W_2d_2$, where W_1 and W_2 are the two weights and d_1 and d_2 are their distances from the point of support.

If the boys are 12 ft. apart and they weigh 100 lb. and 75 lb., respectively, how far from the point of support is the 100-lb. boy?

40. Two boys carry the ends of a pole from which a 175-lb. load is suspended. If the load is supported at a point 3 ft. from one boy and 2 ft. from the other, how much does each boy carry (neglecting the weight of the pole)? NOTE: The amount lifted by one boy, times his distance from the load, equals the amount lifted by the other boy, times his distance from the load. Also, they lift amounts totaling 175 lb., since that is the weight of the load.

41. In the preceding problem, how far from one boy should the load be placed in order for him to carry 125 lb.?

F. Investment Problems

42. A man invests \$2,000 at 2% interest. How much must he invest at 5% in order to make 3% on his total investment?

43. A man invests \$2,400 more at 3% than he invests at 5%. If the total interest is \$360 per annum, how much is invested at each rate?

44. A man invests twice as much as 3% as he invests at 5%. If the amounts were interchanged, his annual income would be increased by \$84. How much does he invest at each rate?

45. A man invests \$5,000 at one rate and \$2,000 at a rate 1% higher. If his total annual income from these investments is \$160, what are the two rates of interest?

46. A man invests \$30,000, part at 2%, part at 3%, and part at 4%. His income from the investments is \$820 per annum. If he doubled the 2% investment, his income would be \$1,020 instead of \$820. Find the amounts invested at the three rates.

G. Digit Problems

NOTE: The number 678 consists of three digits, the units' digit (8), the tens' digit (7), and the hundreds' digit (6). They are given these names because the value of the digit 6 in the number 678 is 6×100, the value of the digit 7 in the number 678 is 7×10, and the value of the digit 8 is 8×1; *i.e.*, the total value of the number is $6(100) + 7(10) + 8$.

47. In a certain two-digit number, the units' digit is 3 less than the tens' digit. The number has a value 26 times that of the units' digit. Find the number; *i.e.*, determine the two digits.

48. Two numbers are written with the same pair of digits. The sum

of the two digits is 12, and the difference of the two *numbers* (not the digits) is 36. Find the digits.

49. The sum of the digits of a two-digit number is 11. If the digits are reversed, the resulting number is 27 more than the original number. What are the digits?

50. The units' digit of a two-digit number is 5 more than the tens' digit. If 72 is added to the number, the result is eleven times the sum of the two digits. Find the digits.

51. A number consists of three digits whose sum is 16. The units' digit is twice the hundreds' digit, and the number is increased by 297 when the order of the digits is reversed. Find the three digits.

41. Type Problems. The following solved problems are intended for reference.

Value Problem. A dealer buys 24 doz. bananas at 25 cents a dozen, then sells them at 55 cents a dozen. Because some of them spoil, his profit is only \$6.10. How many spoil?

Solution: If x is the number of dozens that spoil, the dealer sells $(24 - x)$ dozen at 55¢. This income equals the cost 24×25 (cents) plus the profit 610 (cents). Thus,

$$\begin{aligned} 55(24 - x) &= 600 + 610 \\ 55 \times 24 - 55x &= 1{,}210 \\ 55x &= 1{,}320 - 1{,}210 = 110 \\ x &= 2 \text{ (dozens spoiled)} \end{aligned}$$

Age Problem. A boy is one-third as old as his uncle, and 6 years ago he was one-fifth as old as his uncle was. What are their ages?

Solution: Let x be the age of the boy and $3x$ that of his uncle. Six years ago, the age of the boy was $x - 6$ years and that of his uncle was $3x - 6$ years. By the conditions of the problem,

$$\begin{aligned} 3x - 6 &= 5(x - 6) \\ 3x - 6 &= 5x - 30 \\ 24 &= 2x \\ x &= 12 \end{aligned}$$

Therefore the boy is 12 years old, and his uncle is 36.

Motion Problem

Example 1. An airplane has gas enough for 12 hr. of flight. If its air speed is 180 m.p.h., how far can it go against a 50-m.p.h. wind and get back in the 12 hr.?

Solution: The desired distance is

$$D = (180 - 50)T,$$

where T is the time taken for the outgoing trip.

Also, $$D = (180 + 50)(12 - T)$$

since $12 - T$ is the time taken for the return. The two equations become

$$D = 130T \tag{1}$$
$$D = 2{,}760 - 230T \tag{2}$$

From (1), $$T = \frac{D}{130}$$

Substituting in (2),

$$D = 2{,}760 - \frac{230D}{130}$$
$$D + 1.77D = 2{,}760$$

Then $$D = \frac{2760}{2.77} = 996 \text{ miles, approximately}$$

One could have eliminated D and solved for T, then used T to compute D. This procedure is simpler because D can be eliminated very easily.

Example 2. A plane flies 1,200 miles against a head wind in 8 hr. and returns in 6 hr. Determine the air speed of the plane.

Solution: Let S_p be the speed of the plane. The speed of the wind is unknown and cannot easily be represented in terms of S_p; hence we shall call it S_w.

Then $$1{,}200 = 8(S_p - S_w) = 8S_p - 8S_w \tag{1}$$
and also $$1{,}200 = 6(S_p + S_w) = 6S_p + 6S_w \tag{2}$$
Multiplying (1) by 3, $$3{,}600 = 24S_p - 24S_w$$
Multiplying (2) by 4, $$4{,}800 = 24S_p + 24S_w$$
Adding, $$8{,}400 = 48S_p$$

$$S_p = \frac{8{,}400}{48} = 175 \text{ m.p.h.}$$

Mixture Problems

Example 1. Refer to Example 1, page 66.

Example 2. A goldsmith wishes to make 300 grams of an alloy that is 80% gold. He has stock alloys that are 55% and 95% gold. How much of each should he use?

Solution: Let x be the amount of 55% alloy used, and let $300 - x$ be the amount of 95% alloy. A hidden unknown is the amount of gold in the final mixture. The final mixture contains $.80 \times 300 = 240$ grams

of gold. This equals $.55x + .95(300 - x)$, the sum of the amounts of gold in the portions mixed.

Then
$$\begin{aligned}
240 &= .55x + .95(300 - x)\\
240 &= .55x + 285 - .95x\\
240 - 285 &= .55x - .95x\\
-45 &= -.4x\\
x = \frac{45}{.4} &= 112.5 \text{ grams of } 55\% \text{ alloy}\\
300 - x &= 187.5 \text{ grams of } 95\% \text{ alloy}
\end{aligned}$$

Investment Problem. A man invests \$2,500 more at 3% than he invests at 4%. If his annual income from these investments is 3.3% of his total investment, how much does he invest at each rate?

Solution: Let x be the amount at 4% and $x + 2{,}500$ that at 3%.

Then
$$\begin{aligned}
.04x + .03(x + 2{,}500) &= .033(x + x + 2{,}500)\\
.04x + .03x + 75 &= .066x + 82.5\\
.07x - .066x &= 82.5 - 75\\
.004x &= 7.5\\
x &= \frac{7.5}{.004} = \$1{,}875 \text{ (at 4\%)}
\end{aligned}$$

and
$$1{,}875 + 2{,}500 = \$4{,}375 \text{ (at 3\%)}$$

Digit Problem. The tens' digit of a number is 5 more than the units' digit. If 35 is added to the number, the result is 1 less than four times the number with the digits reversed. Find the number.

Solution: Let x be the units' digit; then $x + 5$ is the tens' digit. The value of the number is $10(x + 5) + x$, while its value with the digits reversed is $10x + x + 5$. By the conditions of the problem, then,

$$\begin{aligned}
35 + 10(x + 5) + x &= 4(10x + x + 5) - 1\\
35 + 10x + 50 + x &= 40x + 4x + 20 - 1\\
11x + 85 &= 44x + 19\\
66 &= 33x\\
x &= 2, \text{ the units' digit}
\end{aligned}$$

The tens' digit is $x + 5 = 7$

and the number is 72.

REVIEW QUESTIONS

1. In solving a stated problem in which there are two unknowns, how may one obtain a single equation in one unknown without first writing two equations and eliminating one unknown?

2. Describe a situation in which the preceding method is of particular advantage.

3. What should be done when the preceding method cannot be applied, or is difficult to use?

4. Is Prob. 10, page 68, a problem in one, two, or three unknowns? Explain.

5. Is Prob. 24, page 69, a problem in one or two unknowns?

6. What is a hidden unknown?

7. When a problem involves two unknowns, only one of which is to be evaluated, which unknown should you attempt to express in terms of the other?

Probably the most important of all study habits is that of *stopping to think*. The person who automatically pauses in his reading, whenever the meaning is not absolutely clear to him, or whenever the writer suggests something for him to think about (or to do), is the person who "sweeps clean" as he studies. In the first five chapters of this book, the double asterisk** has been used as a reminder to assist the student in forming this very desirable habit. It is now time for him to proceed without the help of such a reminder; hence the double asterisk will be omitted from Chap. 6 and will be used only infrequently in the remaining chapters. The student should resolve to perform every operation suggested in the body of the text and to insist on understanding each point before he leaves it.

CHAPTER 6

GRAPHICAL METHODS

Graphical representation is an important aid in establishing a fundamental understanding of the properties of equations and in giving them visual significance. It is also a powerful tool in solving special types of problems and in recording and interpreting data.

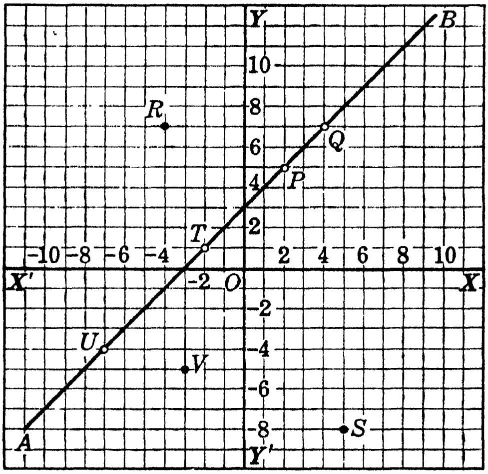


FIG. 4.—Graph of the equation $y = x + 3$.

42. Relation between an Equation and Its Graph. Consider the equation $y = x + 3$. This equation is satisfied by an unlimited number of *pairs* of values of x and y, such as $x = 2$ and $y = 5$, $x = 4$ and $y = 7$, $x = -8$ and $y = -5$, etc. When this equation is represented by a graph, each pair of values of x and y that satisfies the equation is represented by a point on the graph.

In Fig. 4, the graph of the equation $y = x + 3$ is the line AB. In order to see how each point on the graph represents a pair of values of x and y that satisfies the equation, consider the point P on the line AB and answer the following questions:

1. How far (or how many units) to the right of the *origin* O is the point P? (This is the value of x represented by the point P.)

2. How far *above* the origin is the point P? (This is the value of y for the point P.)

From your answers to these questions, you know that the point P represents the pair of values $x = 2$ and $y = 5$. Do these values satisfy the equation $y = x + 3$? Now determine the values of x and y represented by the point Q, and see if they also satisfy the equation.

In general, points to the *left* of the origin represent negative values of x, since left is opposite to right. For example, consider the point R, which represents $x = -4$ and $y = 7$.

Points *below* the origin represent *negative* values of y. Consider the point S, for which $x = 5$ and $y = -8$. Observe that the points R and S are *not* on the line AB. Check the values of x and y which these points represent to see if they satisfy the equation $y = x + 3$.

The complete graph of an equation includes *all* the points representing pairs of values of x and y that satisfy it, *and it includes no other points*. Determine the values of x and y for the points T, U, V; check them to see if they satisfy the equation $y = x + 3$.

43. The Coordinate System. The values of x and y represented by the point P in Fig. 4 are called its x and y *coordinates*. They are usually written in parentheses in the form (x, y). For example, the point $(4, 7)$ represents $x = 4$ and $y = 7$ and is the point Q in Fig. 4. Find the points $(2, 9)$, $(-4, 8)$, and $(-10, -7)$ in Fig. 4. From their positions, would you expect any of them to satisfy the equation $y = x + 3$? Check this.

The line XX' in Fig. 5 is called the x axis, and the line YY' is called the y axis; together they are referred to as the *coordinate axes*. They divide the plane of the coordinate system into the four quadrants labeled I, II, III, IV. For points in quadrant I, x and y are positive; for points in quadrant II, x is negative and y is positive; for points in quadrant III, both x and y are negative; and for points in quadrant IV, x is positive and y is negative. The x coordinate of a point is often called the *abscissa* of the point, and the y coordinate is called the *ordinate* of the point.

EXERCISES (43)

1. Write the coordinates of the points A, B, C, and D in Fig. 5, recording them in the form (x, y).

2. Write the coordinates of the points E, F, G, and H in Fig. 5, recording them in the form (x, y).

3. On a piece of squared paper, draw a pair of coordinate axes XX' and YY'. Locate the points (3, 1) and (−3, 10), and connect

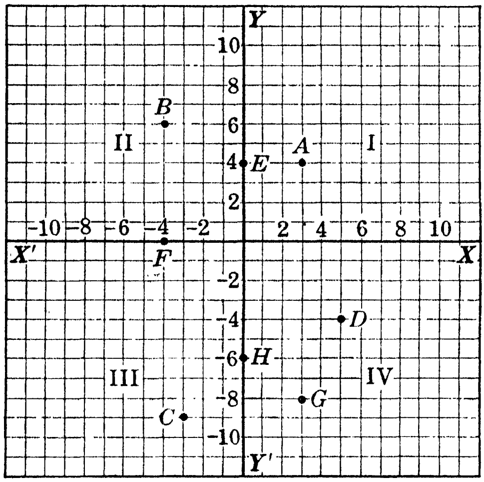


FIG. 5.—The coordinate system.

them with a straight line. Label this line AB. What point on this line has the x coordinate -1? Which point has the x coordinate 5?

4. On the same *set of axes* used in Exercise 3, locate the points (−8, 2) and (−2, −4), and connect them with a straight line labeled CD. At what point does the line CD cross the x axis? At what point does it cross the y axis? (Extend the line if necessary.)

5. On the same set of axes used in Exercises 3 and 4, draw a line from the point (−7, −5) to the point (−3, −3). Label this line EF. At what point does the line EF intersect the line CD of Exercise 4? Where does it intersect the line AB of Exercise 3?

6. At what point does the line EF cross the y axis?

7. At what point does the line EF cross the x axis?

8. Determine the coordinates of the point midway between (2, −2) and (6, −10).

9. If the line between the points (2, −2) and (6, −10) of Exercise 8 is extended, at what point does it cross the x axis? The y axis?

10. Test two points on the line CD (of Exercise 4) to see if they satisfy the equation $y = x - 6$.

44. Linear Equations. A first-degree equation containing two unknowns is called a *linear* equation because its graph is a

straight *line.* Let us now consider the method of *plotting* the graph of a linear equation.

Example 1. Plot the graph of the equation $x - y = 3$.

Solution: First solve the equation for one unknown in terms of the other. In this case, let us solve for y, obtaining $y = x - 3$.

Now choose a value for x, say 10, and find the corresponding value of y from the equation $y = x - 3$. Determine this value of y, and then use it with $x = 10$ to locate a point in Fig. 6. Next choose another value of x, say 5, and repeat the process, locating a second point. Use $x = 0$ and $x = -5$ to locate two more points in the same way.

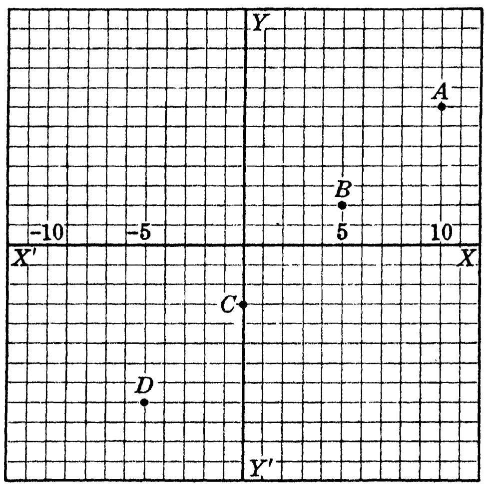


FIG. 6.—Four points on the graph of $x - y = 3$.

You have now located the four points A, B, C, and D in Fig. 6. In order to complete the graph of the equation $x - y = 3$, or $y = x - 3$, it is necessary only to draw a straight line through the points A, B, C, D, since *any point whose coordinates satisfy the equation will be on this line.* Although two points are sufficient to determine a straight line, a third point should be plotted as a check.

In graphing an equation, it is usually best to begin by locating the points where its graph crosses the x and y axes. This is done by choosing $y = 0$ to locate the point where the graph crosses the x axis, then choosing $x = 0$ to locate the point where the graph crosses the y axis. The distances from the origin to these points are called the x and y intercepts, respectively, of the graph.

Example 2. Plot the graph of $2x + y = 4$.

Solution: Let $y = 0$, and solve for x, which is 2. This shows that the desired graph passes through the point (2, 0), or that its x intercept is 2. Check this by referring to Fig. 7. Now let $x = 0$, for which $y = 4$. This shows that the desired graph passes through the point (0, 4), as shown in Fig. 7, or that its y intercept is 4. These two points are sufficient to determine the line, but a third point will be determined

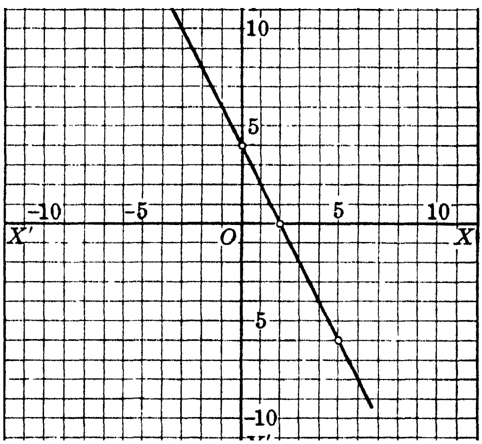


FIG. 7.—Graph of $2x + y = 4$.

as a check on the accuracy of the others. Let $x = 5$; then $2x + y = 4$ becomes

$$10 + y = 4 \qquad \text{or} \qquad y = -6$$

This gives the point (5, −6). Check this point in Fig. 7.

A linear equation in which one of the unknowns is missing is represented graphically by a line parallel to one of the axes.

Example 3. Plot the graph of $y - 7 = 0$.

Solution: This equation is equivalent to $y = 7$; hence its graph includes all points for which y is 7, such as (0, 7), (2, 7), (−5, 7), etc. Refer to a set of coordinate axes, say, in Fig. 7, and observe that such points are on the horizontal line that is seven units above the x axis.

EXERCISES (44)

Plot the graph of each of the following equations. Locate three points in each case.

1. $x + y = 10$	2. $2x - y = 7$	3. $4x - 2y = 4$
4. $3x - y = 6$	5. $y = x + 6$	6. $3x - y = 1$

7. $2x - y = 0$	8. $x - 2 = 0$	9. $2x + y = 8$
10. $x - y = 0$	11. $x + y = 0$	12. $2x - 3y = 9$
13. $y + 3 = 0$	14. $2x - 3y = 7$	15. $x - 3y = 4$

45. Graphical Solution of a Pair of Linear Equations. Refer to Fig. 8, in which the graphs of the two equations

$$2x - y = 4 \quad (1)$$
$$x - y = 1 \quad (2)$$

are plotted on the same set of axes. Since their point of intersection (3, 2) is on *both* graphs, it must satisfy *both* equations. It is, therefore, a *solution* of the *two* equations. Check this by solving the equations (1) and (2) by the methods used in Chap. 4.

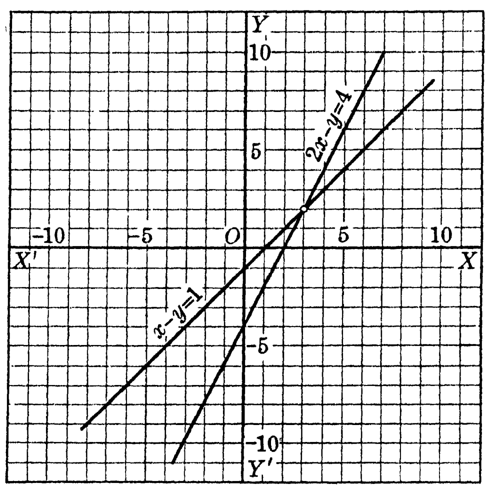


FIG. 8.—Graphical solution of a pair of linear equations.

If two linear equations are not equivalent, their graphs either (1) intersect in a point or (2) are parallel. If their graphs intersect, they are said to be *independent* or *simultaneous* equations. If their graphs do not intersect, two equations are said to be *inconsistent*. Graphs of independent and inconsistent pairs of equations are shown in Fig. 9. Inconsistent equations, since their graphs contain no points in common, are not satisfied simultaneously by any pair of values for x and y. This means that such pairs of equations do not yield a solution. For example, consider $y = 2x + 2$ and $y = 2x - 4$. There is no way of elim-

inating either x or y without eliminating the other; hence there is no pair of values of x and y that satisfies both of these equations. Observe that inconsistent equations are equivalent *except for the constant term*. Examples are $2x - y = 8$ and $2x - y = 12$, $2x - y = 8$ and $4x - 2y = 12$, $3x - 2y = 9$ and $9x - 6y = 18$.

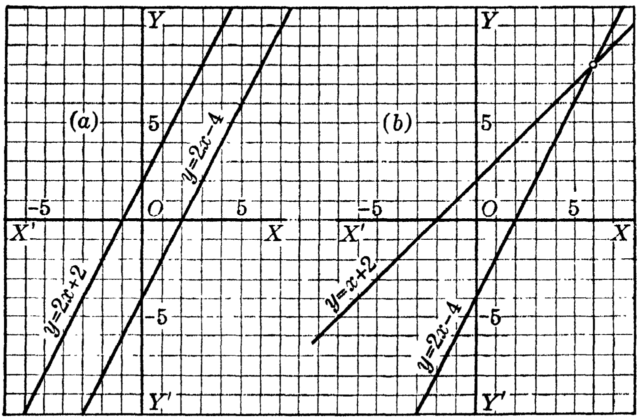


FIG. 9.—Graphs of (*a*) inconsistent and (*b*) independent equations.

EXERCISES (45)

Solve the following pairs of equations by the graphical method. Check their solutions by substitution.

1. $x + y = 5$ $y - x = 3$	2. $x - y = 2$ $y = 4 - x$	3. $2x - y = 2$ $x - 2y = -11$
4. $x + 2y = 14$ $2x + y = 13$	5. $x + y = 5$ $5x - 2y = 4$	6. $y = x - 2$ $x = 4y - 4$
7. $2x - y = 7$ $x + y = 8$	8. $3x - 4y = 9$ $2x + y = 28$	9. $7x - 4y = 20$ $6x - 2y = 10$
10. $2x + y = 8$ $x - y = 7$	11. $2x - y = 4$ $4x - 2y = 12$	12. $2x + y = 6$ $2x + 6 = y$

Solve the following graphically, estimating the resulting values of x and y to the nearest tenth of a unit.

13. $x + 3y = 4$ $4x - 3y = 11$	14. $7x + 3y = 6$ $x + y = 4$	15. $x - y = 5$ $5x + 4y = 12$
16. $3x + 2y = 15$ $2x - 3y = 3$	17. $.5x + y = 2$ $x - y = -4$	18. $5x + 2y = 4$ $5x - 2y = 2$

46. Constants and Variables.

Suppose that Jim and Ronnie, who are learning elementary arithmetic, play the following game: When Jim names a number, Ronnie

must quickly multiply it by 2 and add 5, naming the result. Thus, if Jim says, "Seven," Ronnie must say, "Nineteen"; or, if Jim names 12, Ronnie must name 29.

If x represents the number Jim names, and y the corresponding number that Ronnie must name, the relationship involved in the game can be expressed as

$$y = 2x + 5 \tag{1}$$

In this equation, x and y are *variable* numbers; *i.e.*, they are numbers that can be varied. The word *variable* is also used as a noun; hence x and y are spoken of as *variables*. A variable is a number (usually represented by a letter) whose value is not restricted from varying in a given situation.

In this game, Jim is the "master," and Ronnie is the "slave," for Jim can choose numbers at will but Ronnie has no choice. In other words, y is a *dependent* variable, since its value depends on that of x. The variable whose value can be chosen at will (x in this case) is called the *independent* variable.

Now let us solve (1) for x, obtaining

$$x = \frac{y - 5}{2} \tag{2}$$

This describes a game in which Ronnie is the master and Jim is the slave, but the relationship between the numbers (x and y) which they name is the same. Thus, if Ronnie names 17, Jim must subtract 5, divide by 2, and name the result 6. In this game, x is the dependent variable and y is the independent variable.

Now suppose that Jim and Ronnie wish to change the game, in order to learn other combinations. They could replace 2 by 3 and 5 by 4, so that Ronnie multiplies Jim's number by 3 and adds 4.

Then
$$y = 3x + 4$$

Any such game is described by the general relationship

$$y = ax + b$$

in which a and b are *arbitrary constants*, numbers whose values can be chosen at will for a particular game (or situation), but whose values *remain unchanged* throughout that particular game.

The area of a rectangle is expressed by the relation $A = wl$, where w and l are the width and length of the rectangle. With your eyes closed, imagine a rectangle 2 in. wide and 3 in. long, so that its area is 6 sq. in. Now, imagine the width as constant at 2 in. and the length as increasing slowly from 3 to 10 in. The area, of course, depends on the length and increases simultaneously to 20 sq. in. Be sure that you imagine a *growing* rectangle whose width is constant at the arbitrary value of 2 in., which was chosen in the beginning. Now, choose a new width, say, 5 in., and let the length start at 10 in., increasing slowly to 15 in. The area, of course, will increase from 50 to 75 sq. in. Observe that we are thinking of l as the independent variable, of A as the dependent variable which depends on l, and of w as an *arbitrary constant*, which we choose each time but which does not change while l and A are changing.

It is important to be able to visualize quantities as variable, or capable of being changed. To go one step further, let us think of *both* w and l as variables. Let $w = 5$ and $l = 10$, so that $A = 50$. Now let l increase slowly to 15 in., but let w increase *at the same time* to 9 in. As w and l *both* change, A increases from 50 to 135 sq. in., its value at any time depending on the values of *both* w and l, which are variables. Be sure that you close your eyes and "see" a rectangle whose width, length, and area are increasing slowly *at the same time.*

EXERCISES (46)

Instead of working exercises and problems associated directly with the material in this section, it is suggested that you go back to Chap. 1 and check up on your various mental abilities. See if your powers of visualization, etc., are still up to par, and have a try at some of the exercises and problems in Chap. 1. You will have forgotten how to do the problem of the cannibals—see how long it takes you. Take this opportunity to make yourself "mentally fit" before going on with algebra.

47. Solution of Equations in One Unknown. Consider the equation $3x - 7 = 0$. This equation is so easy to solve by the methods already learned that it would be pointless to consider a graphical method for solving it. For the present, therefore, we will pass on to equations of higher than the first degree.

Example 1. Consider the second-degree, or *quadratic*, equation

$$z^2 - 5z + 4 = 0$$

To *solve* this equation is to find the value (or values) of z for which the equation is satisfied, *i.e.*, for which the value of the expression $z^2 - 5z + 4$ is zero. Let us attempt to find these values (there are two) by systematic trial.

If $z = 0$	then	$z^2 - 5z + 4$	equals	4
If $z = 1$	then	$z^2 - 5z + 4$	equals	0 (root)
If $z = 2$	then	$z^2 - 5z + 4$	equals	-2
If $z = 3$	then	$z^2 - 5z + 4$	equals	-2
If $z = 4$	then	$z^2 - 5z + 4$	equals	0 (root)
If $z = 5$	then	$z^2 - 5z + 4$	equals	4

There is no point in going further, since we have found the two values of z for which the expression $z^2 - 5z + 4$ has the value zero. These values $z = 1$ and $z = 4$ are the roots of the equation $z^2 - 5z + 4 = 0$.

In assigning different values to z (in order to find the values for which it satisfies the equation $z^2 - 5z + 4 = 0$), we are thinking of it as a *variable* number, which can have any desired value. Thus z may have any value, in general, but only certain of those values satisfy the desired equation. Observe that the value of the quantity $z^2 - 5z + 4$ depends on the value chosen for z; therefore for every value of z there is a value for $z^2 - 5z + 4$. This means that $z^2 - 5z + 4$ is a *function* of z.

> ***If a quantity depends on a variable, say, z, in such a way that for every value of z there is at least one value of the quantity, it is said to be a function of z.***

Examples:

	$2w - 8$	is a function of w
	$3r - 5$	is a function of r
	$s^2 - 25s - 3$	is a function of s
	$x^3 - 3x^2 + 2x + 1$	is a function of x
	$2y - 7$	is a function of y

Although functions will be described in a more general sense in the next section, it is important just now to regard a function as

an expression containing a variable, and to call it a function of that variable.

Example 2. Solve the equation $w^2 - 4w + 2 = 0$

Solution: In order to solve this equation, we must find the values of w for which the *function* $w^2 - 4w + 2$ has the value zero. It happens that these values of w are not integers, which makes the process of systematic trial somewhat longer. For this reason, we shall set up the trial values of w and the corresponding values of the function in a table.

If w is	0	1	2	3	4
then $w^2 - 4w + 2$ equals	2	-1	-2	-1	2

At this point we should stop, for we can see that the function

$$w^2 - 4w + 2$$

has the value zero for some value of w *between* $w = 0$ and $w = 1$, and also somewhere between $w = 3$ and $w = 4$. Check this by means of the table, and observe that the first value of w is apparently nearer $w = 1$ than $w = 0$, while the second is apparently nearer $w = 3$ than $w = 4$.

We now make up a table for values in these ranges (check a few of the values).

w	.5	.6	.59	.58	3.5	3.4	3.41	3.42
$w^2 - 4w + 2$	.25	$-.04$	$-.0119$	$+.0166$	.25	$-.04$	$-.0119$	$+.0164$

Refer to the values listed in the table, and verify the following statements:

1. The root between 0 and 1 is also between .5 and .6. Going a step further, we can see that it is between .59 and .58. Suppose we say that this root is *approximately* .585.

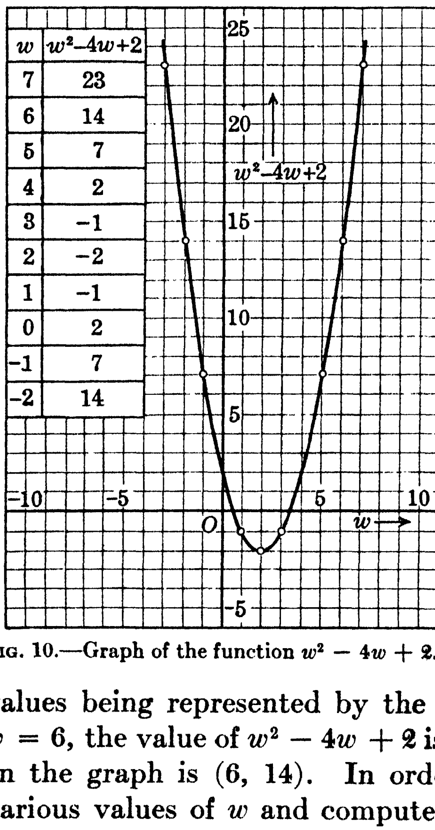
2. The root between 3 and 4 is also between 3.5 and 3.4, and finally between 3.41 and 3.42. Suppose we say that this root is approximately 3.415.

In recording .585 and 3.415 as approximate values of the roots of the equation $w^2 - 4w + 2 = 0$, we should write $w \doteq .585$ and $w \doteq 3.415$. The symbol $\doteq$ is read, "equals approximately . . . ." It should be used whenever the relation to be indicated is not an *exact* equality.

The process of finding more accurate values of these roots can be carried on indefinitely (to more and more decimal places); for their values

cannot be expressed *exactly* by any number of decimal places. We can determine them accurately enough for practical purposes, however, without a great deal of difficulty.

48. Graphical Solution. The disadvantages of the preceding method are apparent; for it is necessary to test many values of the variable in order to determine the roots with any degree of precision. Also, it is not always possible to make "guesses" that turn out to be so near the desired values as those listed in this example. The method of systematic trial can be improved a great deal by means of graphical representation, as will now be demonstrated.



w	w^2-4w+2
7	23
6	14
5	7
4	2
3	-1
2	-2
1	-1
0	2
-1	7
-2	14

FIG. 10.—Graph of the function $w^2 - 4w + 2$.

The graph of the function $w^2 - 4w + 2$ is shown in Fig. 10. The y distance of each point on this graph represents a value of the function $w^2 - 4w + 2$, and the x distance of the same point represents the corresponding value of w. For example, when $w = 4$, the value of $w^2 - 4w + 2$ is 2, these values being represented by the point (4, 2) on the graph. At $w = 6$, the value of $w^2 - 4w + 2$ is 14, and the corresponding point on the graph is (6, 14). In order to plot the graph, we choose various values of w and compute the corresponding values of the function $w^2 - 4w + 2$, recording them as in the table in Fig. 10. Then we plot the points corresponding to these pairs of values and connect them by a smooth curve, as shown.

Consider the information that is supplied by the graph of the function $w^2 - 4w + 2$. (1) It gives a complete, visualized impression of the way in which $w^2 - 4w + 2$ changes as w is changed. (2) It indicates that $w^2 - 4w + 2$ has the value *zero* at the two points for which $w \doteq .6$ and $w \doteq 3.4$, respectively, showing that these values are the (approximate) roots of the

equation $w^2 - 4w + 2 = 0$. (Refer to the graph to check these values.)

If very accurate values for the roots are desired, they can be obtained by plotting sections of the graph (near the desired values of w) on a large scale, as shown in Fig. 11. Such small sections of the graph appear to be practically straight, making it easy to

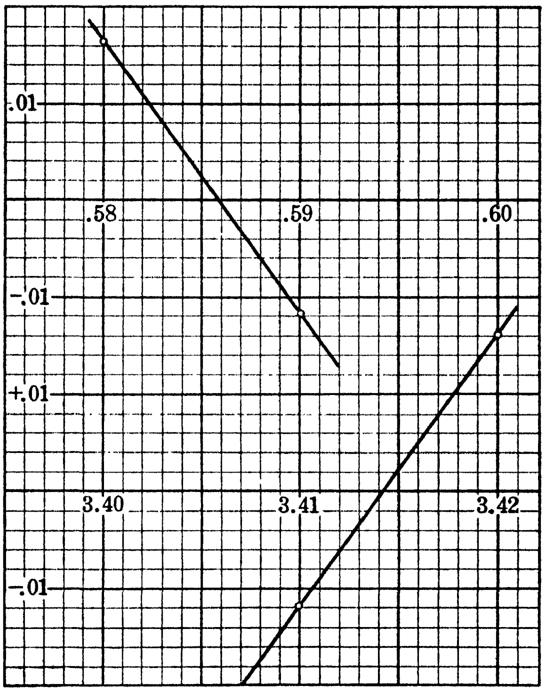


FIG. 11.—Accurate graphical solution.

obtain very accurate values without plotting many points. Read the values of the roots from Fig. 11, and compare them with the results obtained by trial.

In summary, the graphical solution of the equation

$$w^2 - 4w + 2 = 0$$

consists of plotting a graph of the function $w^2 - 4w + 2$ in order to find its *zeros*, or the values of w for which it equals zero. It must be understood that this is *not* plotting the graph of the *equation* $w^2 - 4w + 2 = 0$.

Since the function $w^2 - 4w + 2$ is of second degree, it is called a second-degree or *quadratic* function, and the equation

$w^2 - 4w + 2 = 0$ is called a *quadratic equation*. Not all quadratic equations have actual solutions. This means that in some cases there is no ordinary number which, when substituted for the variable, will satisfy the equation. Such an equation is

$$x^2 - 4x + 5 = 0$$

which one might attempt to solve by plotting a graph of the function $x^2 - 4x + 5$, as in Fig. 12. For all values of x, the numerical value of the function $x^2 - 4x + 5$ is greater than zero, as is shown by the graph. This means that the *equation* $x^2 - 4x + 5 = 0$ has no solution, in the ordinary sense. *Imaginary* solutions for such equations will be defined later.

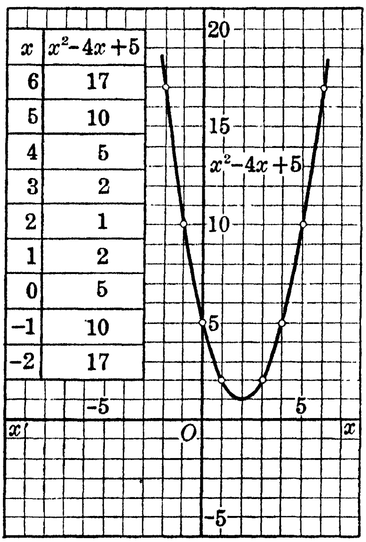


FIG. 12.—Graph of the function $x^2 - 4x + 5$.

The curve obtained by plotting a quadratic function is called a *parabola*, the practical applications of which are numerous. For example, the path of a bullet is theoretically a parabola, though the effect of air resistance distorts it from a true parabola. The most efficient shape for a reflector used in an automobile headlamp is that of a parabola rotated on its axis; the paths of some comets are parabolas.

The general procedure used in graphing a function of one variable, regardless of its degree, is as follows:

1. Choose arbitrary values for the variable.
2. Compute the corresponding values of the function.
3. Plot the pairs of values thus obtained, and connect the points with a smooth curve.
4. Plot extra points in the portions of the graph which are to be examined in detail or which seem to show unusual properties.

Example 1. The graph of the function $2z - 5$ is shown in Fig. 13. The graph of any function of the first degree is a straight line; hence first-degree functions are commonly referred to as *linear* functions. Choose several values of z (say $z = 1$, 4, and 5), and compute the corre-

sponding values of the function $2z - 5$. Use these values to check several points in Fig. 13 to verify that it is actually the graph of the functions $2z - 5$. Now observe the value of z at the point where the graph crosses the horizontal axis. Does this value of z satisfy the equation $2z - 5 = 0$? The graph of the function $2z - 5$ is the same as the graph of the equation $y = 2x - 5$; for in it we represent values of $2z - 5$ by y distances and z values by x distances.

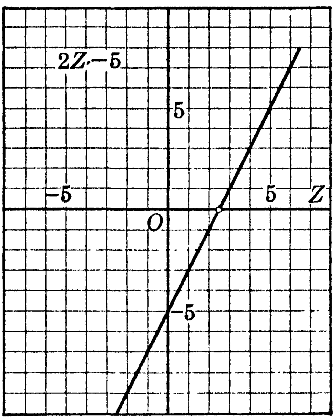


Fig. 13.—Graph of the linear function $2z - 5$.

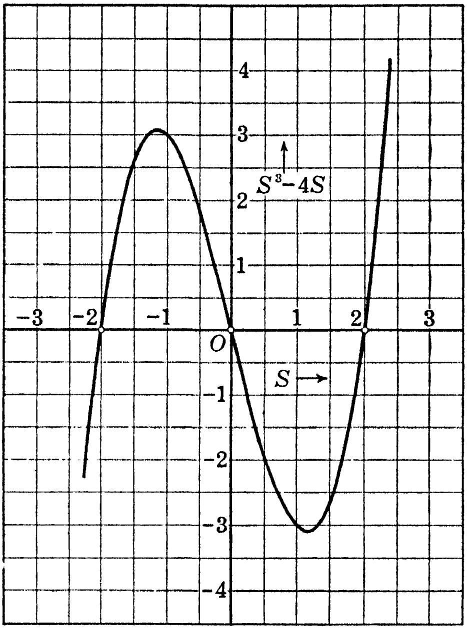


Fig. 14.—Graph of the cubic function $s^3 - 4s$.

Example 2. Solve the equation $s^3 - 4s = 0$.

Solution: Plot a graph of the function $s^3 - 4s$, as in Fig. 14. The function $s^3 - 4s$ has its "zeros" at $s = 0$, $s = 2$, and $s = -2$; hence these are the three roots of the equation $s^3 - 4s = 0$. The equation $s^3 - 4s = 0$ is a third-degree or *cubic* equation, and the quantity $s^3 - 4s$ is called a *cubic function.*

EXERCISES (48)

Plot the following functions:

1. $3w - 8$	2. $2r + 7$	3. $z^2 - 5z + 4$
4. x^2	5. $v^2 - 25$	6. $x^2 - 3x$
7. $v^2 - 7v + 3$	8. $\frac{1}{v}$	9. $w^3 - w$

Use the graphs plotted in the preceding exercises to solve these equations:

10. $3w - 8 = 0$ 11. $z^2 - 5z + 4 = 0$ 12. $v^2 - 7v + 3 = 0$
13. $x^2 - 3x = 0$ 14. $v^2 - 25 = 0$ 15. $w^3 - w = 0$

49. Functions. In the preceding section the student was encouraged to think of a function as an algebraic expression containing a variable. This basic concept will now be related to equations and physical quantities.

Example 1. Consider the equation $2v - w = 8$. For every value of v that we might choose, this equation specifies a corresponding value of w. For example, if $v = 1$, $w = -6$; and if $v = 5$, $w = 2$. This means that the value of w depends on the value chosen for v, or that *w is a function of v*. In many cases it is not enough to know that w is a function of v; *i.e.*, it may be necessary to know *what* function of v it is. Let us solve the equation $2v - w = 8$ for v, obtaining

$$w = 2v - 8$$

In this form the equation shows that w equals the function $2v - 8$. Thus it specifies, not just that w depends on v, but *how* it depends on v. Writing the equation in the form $w = 2v - 8$ is referred to as *expressing w as a function of v*, or expressing w in terms of v.

Now, let us solve the equation $2v - w = 8$ for v, obtaining

$$v = \frac{w + 8}{2}$$

In this form of the equation, v is expressed as a function of w, specifically, as the function $\frac{w + 8}{2}$. For any value of w that we choose, there is a corresponding value $\left(\frac{w + 8}{2}\right)$ of v. Thus, from the equation $2v - w = 8$, we can obtain *either* variable as a function of the other.

In order to represent the equation $2v - w = 8$ by a graph, let us choose values for either w or v and list the corresponding values of the other, as follows:

w	0	-8	2	5	9
v	4	0	-5	$6\frac{1}{2}$	$8\frac{1}{2}$

In order to use these values to plot a graph, it is necessary to decide which variable (w or v) is to be represented on the vertical axis, *i.e.*, by y distance. If values of w are represented by y distance and values of

v by x distance, the graph in Fig. 15 is obtained. On the other hand, if values of v are represented by y distance (and those of w by x distance), the graph in Fig. 16 is obtained. At this point, the student should ask himself what is meant by "the graph of the equation $2v - w = 8$." *Two* graphs have been obtained from this equation: one by plotting w on the vertical axis, the other by plotting v on the vertical axis. To refer to "the graph of the equation $2v - w = 8$" does not specify which of the two possible graphs is meant. For this reason, refer to *the* graph of an

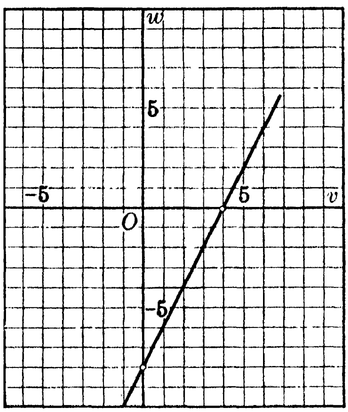


FIG. 15.—Graph of w as a function of v, for $2v - w = 8$.

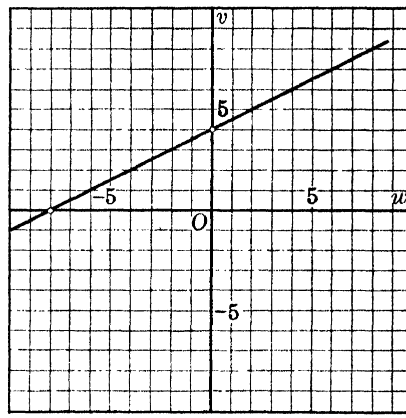


FIG. 16.—Graph of v as a function of w, for $2v - w = 8$.

equation only when the variables in the equation are x and y, in which case it is known that y is to be plotted on the vertical axis.

It was shown earlier that $w = 2v - 8$, or that w is equivalent to the function $2v - 8$. In plotting values of w as y distances, in Fig. 15, we are simply plotting the graph of the function $2v - 8$; *i.e.*, we are plotting *w as the function* $2v - 8$. For this reason, Fig. 15 is referred to as *the graph of w as a function of v*. Similarly, Fig. 16 is the graph of v as a function of w, since it is the graph of the function $\frac{w + 8}{2}$, which equals v. Plotting v on the y axis (and w on the x axis) is called *plotting v as a function of w*.

Example 2. The distance traveled by an automobile at a constant speed of 40 m.p.h. depends on, or is a function of, the number of hours spent in travel. If D is the distance and H is the number of hours, then $D = 40H$.

Figure 17 is the graph of D as a function of H, for $D = 40H$. Observe that at $H = 2$ hr., $D = 80$ miles, etc. In this graph, equal distances along the vertical and horizontal axes do not represent equal numerical

values. The scales are made unequal in order to make the graph easy to read and of convenient size.

Example 3. The parcel-post charge for a package weighing P lb. is $(\frac{1}{2}P + 7\frac{1}{2})$ cents in a certain zone. The charge $\frac{1}{2}P + 7\frac{1}{2}$ is a function of the number of pounds. If C represents the charge, then the equation

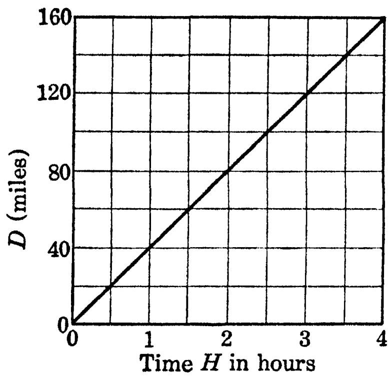


FIG. 17.—Graph of D as a function of H, for $D = 40H$.

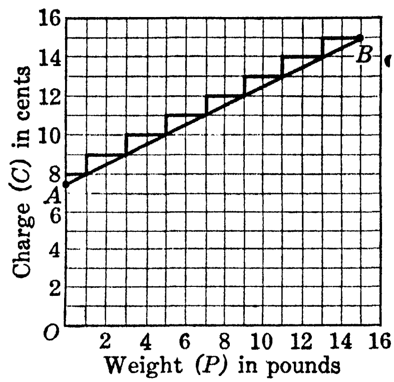


FIG. 18.—The parcel-post function, $\frac{1}{2}P + 7\frac{1}{2}$.

$C = \frac{1}{2}P + 7\frac{1}{2}$ expresses C as a function of P. The graph of C as a function of P, when $C = \frac{1}{2}P + 7\frac{1}{2}$, is the straight line AB in Fig. 18. At the post office, a fraction of a cent counts as a whole cent; hence the actual charge is that shown by the "stair-step" graph in Fig. 18 rather than by the straight line. Observe that the equation $C = \frac{1}{2}P + 7\frac{1}{2}$ does not express the actual charge, because it does not count a fraction of a cent as a whole cent. This is a case in which the graph conveys information that cannot conveniently be put in the equation.

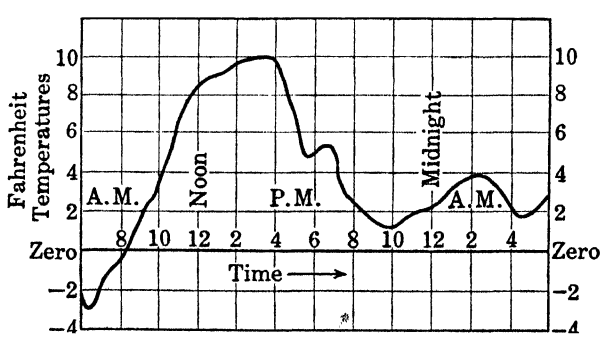


FIG. 19.—Temperature as a function of time, for Boston, Jan. 24, 1945.

A record of the temperature in Boston for a period of 24 hr. is shown in Fig. 19. At any time during the 24 hr., a value of temperature is shown by the graph; hence the graph expresses the

temperature as a function of the time. Now, suppose we wish to say *what* function of time the temperature is or *how* the temperature depends on the time. We can represent temperature by T and time by t, but we cannot write $T = \cdots$, because no algebraic function describes the temperature. We can say, however, that the temperature at a specified place in Boston, on Jan. 24, 1945, is shown as a function of time in Fig. 19. In many cases, a graph is the most effective means of portraying the relationship between two quantities.

EXERCISES (49)

1. If $3x - y = 7$, express y as a function of x, and *underline* the function.
2. If $3x - y = 7$, express x as a function of y.
3. If $2v - v^2 + w - 7 = 0$, express w as a function of v.
4. In the equation $r^2 - 2s + 5r - 6 = 0$, is s a function of r? What function?
5. Give three examples of functions that are physical quantities, such as, "The amount of gasoline consumed by an automobile is a function of the distance traveled."
6. Use Fig. 19 to answer the following questions:

a. What was the temperature at 9 A.M?

b. What was the lowest temperature, and when did it occur?

c. What was the highest temperature?

d. When was the temperature 6° above zero (two answers)?

e. Will the next day probably be colder than the one shown?

50. Graphical Solution of Stated Problems. In general, the graphical method is more laborious than analytic methods of solving problems. For this reason, graphical representation is used primarily as a means of interpreting and understanding other methods of solving problems. There are many cases, however, in which the graphical method is superior to other methods. The following examples illustrate such cases:

Example 1. One pump will fill a certain tank in 12 hr., while another will fill it in 8 hr. How long would be required for the two pumps (together) to fill the tank?

Solution: Problems of this type will be solved analytically in Chap. 8; hence we shall consider now only the graphical solution, shown in Fig. 20. In this graph, the filling of the tank is represented by the vertical distance OA, and time in hours is represented by x distance. The line OB is

drawn to the point B (12 hr.), and the line AC is drawn to the point C (8 hr.). The time required for the two pumps to fill the tank is read at the intersection D, as 4.8 hr. In detail, the line OB indicates the portion of the tank filled by the first pump, as a function of time; and the line AC represents the portion filled by the second pump, as a function of time, *plotted downward* from A. At D, observe that $DE + DF = OA$, showing that the tank is full at that point. An attempt to solve this problem analytically will convince the student that the graphical method sometimes has its advantages.

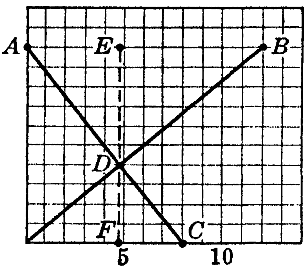


FIG. 20.—Example 1.

Example 2. A bomber is stripped of all accessory equipment and loaded with gasoline for a 3,000-mile nonstop flight to an advanced base in the Pacific. It travels at 200 m.p.h., thus requiring 15 hr. for the trip. Three hours after the bomber leaves, a smaller plane starts along the same route, at 300 m.p.h. Since it has a smaller cruising range, the smaller plane has to stop at a refueling station 1,650 miles from the starting point, in line with the route of the bomber. At what point does the smaller plane overtake the bomber?

Solution: There are three possibilities. The smaller plane may overtake the bomber

1. Before the refueling station is reached.
2. After the refueling station is reached.
3. Not at all.

If we assume Case 1, the bomber takes 3 hr. longer than the smaller plane to travel the distance d to the point where it is overtaken. Thus

$$\frac{d}{200} = \frac{d}{300} + 3 \qquad (1)$$

since the bomber takes $\frac{d}{200}$ hr. and the other plane $\frac{d}{300}$ to travel the distance d. Multiplying each member of (1) by 600,

$$3d = 2d + 1{,}800$$
$$d = 1{,}800 \text{ miles}$$

This result is now seen to be incorrect, since a point 1,800 miles from the starting point is beyond the refueling station, where the second plane is delayed by an hour. Taking the delay into account,

$$\frac{d}{200} = \frac{d}{300} + 3 + 1 \qquad (2)$$

M(600), $\quad 3d = 2d + 2{,}400$

$$d = 2{,}400 \text{ miles}$$

Now let us use the graphical method. In Fig. 21, the line OA represents the progress of the bomber, with the distance traveled represented by y distance, and the time by x distance. Observe that at 1 hr. the distance traveled is 200 miles, at 2 hr. the distance is 400 miles, etc. The path $CDEFG$ represents the progress of the second plane. The portion CD represents its flight at 300 m.p.h. for 1,650 miles; DE represents the delay of 1 hr.; and EFG represents the remaining portion of the flight.

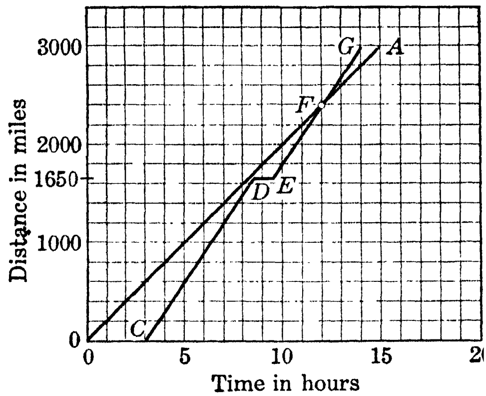


FIG. 21.—Example 2.

The intersection at F indicates that the bomber is overtaken 2,400 miles from the starting point.

Not only is the graphical solution more straightforward in this case, but it gives a complete picture of the situation. Observe, for example, that the second plane is only 50 miles behind the bomber when it has to stop to refuel, and that it completes the flight 1 hr. ahead of the bomber.

Now consider the analytic method again, and observe that one cannot get out of solving equation (1) by guessing (correctly) that the bomber is not overtaken until after the refueling station is passed. The reason is that the bomber *might* be overtaken *twice*, in which case each equation would produce an answer. It is suggested that the student set up a graphical solution for a problem in which the second plane starts only 2 hr. behind the bomber, in order to see the full possibilities of the problem.

PROBLEMS (50)

Solve the following problems graphically:

1. One man can mow a lawn in 2 hr. and 20 min., while his son can do it in 3 hr. How long will it require the two, working together?

2. An airplane can fly from Boston to Chicago in 15 hr. Another can fly from Chicago to Boston in 10 hr. If they leave Boston and Chicago, respectively, at the same time, how soon will they meet?

3. One pump alone can fill a tank in 12 hr. With the aid of another pump, it can fill the same tank in 7 hr. How long would it take the second pump alone?

4. Two pumps empty a tank in 90 min., while with the aid of a third pump they can empty it in 40 min. How long would it require the third pump alone?

5. Two automobiles start toward each other from two towns 280 miles apart. If one travels 30 m.p.h., and the other 40 m.p.h., how soon will they meet?

6. At 8 A.M., Jimmie starts along a highway on a bicycle, traveling 10 m.p.h. After 3 hr., he stops for lunch at a roadside café. He spends half an hour at the café, then resumes his journey. His companion, who starts at 9:30 A.M., travels along the same route at $13\frac{1}{3}$ m.p.h. until he overtakes Jimmie. By how much time did he miss catching Jimmie at the café? How far from the starting point did he finally overtake him?

7. One electrically driven pump can fill a certain tank in 40 hr. Another pump, if it could be operated continuously, could fill the same tank in 36 hr. The second pump, however, is designed for intermittent operation; therefore after each 4 hr. of operation it must be turned off for 1 hr. How long would be required for the two pumps to fill the tank?

8. A through train leaves Chicago. After traveling for 3 hr. at 50 m.p.h., it is delayed for 2 hr. by engine trouble; then it travels on at 10 m.p.h. If its average speed for the entire trip is 20 m.p.h., how far did it travel?

51. Determining the Maximum or Minimum Value of a Function (Optional). Consider this situation: In a plant where sheet-metal sections for an airplane are stamped out and pressed into the proper shape, one of the cutting operations leaves thousands of scrap pieces of thin sheet metal 8 by 12 in. The foreman decides to use these pieces to make small boxes to hold fuses and small replacement parts. The method is illustrated in Fig. 22. A small square, x in. on a side, is cut out of each corner; then the metal is folded along the dotted lines to make a box without a top. The corners are then soldered. The dimensions of the box are length, $12 - 2x$; width,

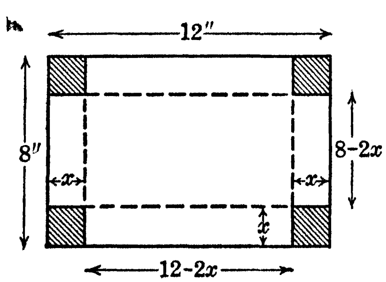


FIG. 22.—Making a rectangle into a box.

$8 - 2x$; depth, x. The volume of the box is

$$V = (12 - 2x)(8 - 2x)x$$

This can be written as

$$V = 4x^3 - 40x^2 + 96x$$

Now, the problem is to determine the size of the squares one should cut out to make the volume V of the box have the greatest possible value. One method of doing this is to plot V as a function of x, noticing from the graph what value of x makes V largest.

The graph of V as a function of x is plotted in Fig. 23. Refer to

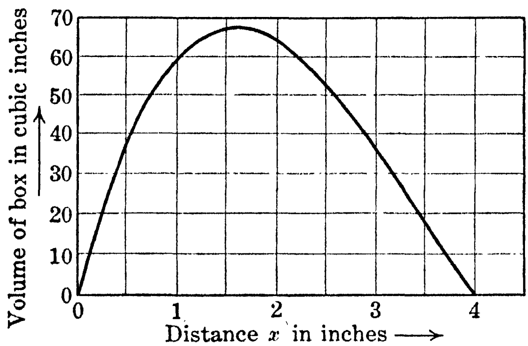


FIG. 23.—Volume ($4x^3 - 40x^2 + 96x$) as a function of x.

this graph, and determine both the maximum possible value of V and the corresponding value of x.

The maximum possible value of V can be determined very accurately from the graph. The corresponding value of x, however, cannot be estimated with such precision, since the curve is at practically the same height for values of x between 1.4 and 1.8 in. From the practical viewpoint, it would be satisfactory to cut out squares 1.5 in. on a side, since V would be only a little below maximum value.

It is not necessary to plot the complete graph in solving such problems. After plotting a few points, one has a good idea of the approximate location of the maximum and can choose points near that position.

EXERCISES (51)

In solving the following problems, plot only a small section of each graph:

1. What is the smallest value of $x^2 - 4x + 5$, and for what value of x does it occur? NOTE: $x^2 - 4x + 5$ is plotted as a function of x in Fig. 12, page 90.

2. Determine the maximum value of $4x - x^2 + 2$.

3. A rancher wishes to fence off a rectangular pasture along a river. What is the greatest area he can enclose with 1 mile of fence, if no fence is necessary on the side next to the river? HINT: If L is the length of the pasture and W is its width, then $A = LW$. Also $L + 2W = 1$ mile (draw a sketch to verify this). Now obtain an equation for A in terms of W, and plot A as a function of W, finding enough points to determine the maximum.

4. Divide the number 22 into two parts such that the sum of their squares is a minimum.

5. What number exceeds its square by the greatest amount?

6. When a ball is thrown vertically into the air, its distance from the ground at any later time (until it strikes the ground) is $st - 16t^2$. In this expression, s is the speed with which the ball is thrown, and t is the time in seconds from the instant it is thrown. If a ball is thrown upward at a speed of 100 ft. per second, how high does it rise?

7. In Prob. 6, how long does the ball remain in the air?

52. Functional Notation (Optional). Consider the equation $w = z^2 - 5z + 2$, in which w is expressed as a function of z. Suppose that it is wished to record several values of z, along with the corresponding values of w. Thus, if $z = 1$, $w = -2$; if $z = 2$, $w = -4$; if $z = 5$, $w = 2$; etc. This information can be recorded much more conveniently by the use of *functional notation*. To write the equation $w = z^2 - 5z + 2$ in functional notation, we record in parentheses, after w, the variable (or variables) on which it depends. Thus we write

$$w(z) = z^2 - 5z + 2$$

which is read, w (of z) equals $z^2 - 5z + 2$. *The parentheses do not indicate multiplication*, but only the fact that it is z on which w depends. This notation makes it possible to record numerical values very simply, as follows:

Instead of "when $z = 1$, $w = -2$," write only $w(1) = -2$
Instead of "when $z = 2$, $w = -4$," write only $w(2) = -4$
Instead of "when $z = 5$, $w = 2$," write only $w(5) = 2$, etc.

In each case, z is merely replaced by its numerical value. Be sure to read: "w(of 1) equals -2," etc., rather than "w times 1," etc.

Consider the relation $z = r^2 - 5rs + 7$, in which z is expressed as a function of the *two* variables r and s. In functional notation,

$$z(r, s) = r^2 - 5rs + 7$$

Compare the use of functional notation with ordinary notation in the following:

Ordinary Notation		Functional Notation
When $r = 1$ and $s = 2$,	$z = -2$	$z(1, 2) = -2$
When $r = 3$ and $s = 4$,	$z = -47$	$z(3, 4) = -47$
When $r = 2$ and $s = 1$,	$z = 1$	$z(2, 1) = 1$
When $r = 2$ and $s = -1$,	$z = 21$	$z(2, -1) = 21$

Functional notation can also be used to show which symbols represent variables and which represent arbitrary constants. In the equation $y = ax + b$, we can be reasonably sure that x and y are variables and a and b constants, because it is customary to use letters near the end of the alphabet to represent variables. In the equation $z = r^2 - 5rs + 7$, however, we cannot be so sure. If it is desired to designate s as an arbitrary constant, rather than a variable, one should write

$$z(r) = r^2 - 5rs + 7$$

This shows that r is the only variable on which z depends in this equation; hence s must be a constant. Likewise, if one writes $w(l, r) = l^2 - 2r + 5$, he is thinking of both l and r as variables. If he writes $w(l) = l^2 - 2r + 5$, r is a constant; or if he writes $w(r) = l^2 - 2r + 5$, l is a constant.

In the equation $y = x^2 + 2x + 1$, it is evident that y is a function of the variable x. The functional notation is of advantage in this case, however, for recording pairs of values of x and y by the method described earlier, *viz.*,

$$y(1) = 4$$
$$y(2) = 9$$
$$y(3) = 16, \text{ etc.}$$

Functional notation has other important uses, which are encountered in more advanced courses.

EXERCISES (52)

1. The formula for the volume of a cylinder is $V = \dfrac{\pi d^2 l}{4}$, where d is the diameter of the cylinder and l is its length. Now write the formula

in a way which shows that you are thinking of the volume of a varying length of pipe of fixed diameter d.

2. Write the formula of Exercise 1 in such a way as to indicate the volume of a fixed length of pipe whose diameter is thought of as varying.

3. Write the formula of Exercise 1 in such a way as to indicate that both d and l are thought of as variables.

4. Use functional notation to express the volumes of three cylinders for which $d = 3$ ft., $l = 7$ ft.; $d = 6$ ft., $l = 10$ ft.; and $d = 5$ ft., $l = 7$ ft., respectively.

5. If $z(x, y) = x^2 + 2xy + 3$, find $z(1, 3)$, $z(4, 0)$, $z(3, -5)$.

6. If $y(x) = 6x + 17$, find $y(0)$, $y(1)$, $y(2)$, $y(-2)$.

7. If $z(x) = 2x^2 - 3x + 7$, find $z(0)$, $z(2)$, $z(a)$.

8. If $D(t) = t^3 - 3t^2 + 7t + 5$, find $D(0)$, $D(-1)$, $D(2)$.

9. If $3a - b = 7$, find b as a function of a, and write the result in functional notation.

10. If $x^2 - 2xz + 2y - 4 = 0$, express y as a function of x and z, writing the result in functional notation.

11. If $xy = 25$, express y as a function of x, and write the result in functional notation.

12. If $g(x, y) = \dfrac{2x + y + 4}{x - y - 2}$, find $g(0, 0)$, $g(1, 2)$, $g(-1, -2)$.

REVIEW QUESTIONS

1. Suppose you were asked to solve the following pair of equations:

$$\begin{aligned} 2x - y &= 7 \\ x + y &= 2 \end{aligned}$$

Can it be done? Why or why not?

2. Suppose you were asked to solve the following pairs of equations:

$$\begin{aligned} 2x - y &= 4 \qquad & 2x - y &= 3 \\ 4x - 2y &= 4 \qquad & 6x - 3y &= 9 \end{aligned}$$

Can it be done in these cases? Why or why not?

3. How does one solve the equation $s^2 - 4s + 2 = 0$ graphically?

4. What is meant by the "zeros" of a function, and how are they related to the solution of an equation?

CHAPTER 7

FACTORING

If an algebraic expression is the product of two or more quantities, those quantities are called *factors* of the expression. *Factoring* an expression is the process of determining its factors. Though not an end in itself, factoring is very important in the solution of equations of higher than the first degree, and in the handling of fractions.

53. Applications of Factoring. Consider the equation

$$x^2 - 5x + 4 = 0$$

In the preceding chapter, equations of this type were solved by plotting the expression equated to zero ($x^2 - 5x + 4$) as a function of x and finding its "zeros." In many cases the roots of a quadratic equation can be obtained by a much simpler process, which will now be demonstrated.

The expression $x^2 - 5x + 4$ can be written in the *factored form* $(x - 4)(x - 1)$. Verify this by multiplying $x - 4$ by $x - 1$. Now observe that if *either* $x - 4$ or $x - 1$ equals zero, their product equals zero. This proves that $x^2 - 5x + 4$ equals zero when $x - 4 = 0$ and also when $x - 1 = 0$.

$$\text{From } x - 4 = 0, \quad x = 4 \qquad \text{From } x - 1 = 0, \quad x = 1$$

Thus $x = 4$ and $x = 1$ are the roots of the original equation $x^2 - 5x + 4 = 0$.

Instead of solving the equation $x^2 - 5x + 4 = 0$ directly, we have solved the two first-degree equations $x - 4 = 0$ and $x - 1 = 0$. These equations are obtained by equating the factors of the expression $x^2 - 5x + 4$ to zero. This convenient method of solving equations is one of the two most important applications of factoring: the other is its use in the handling of fractions and fractional equations. The following study of methods of factoring presents the material essential to these two applications of factoring.

54. Factors of a Monomial. The factors of a monomial are evident in the written form of the monomial. For example, the monomial $6x^2y^3z$ has the factors 6, x^2, y^3 and z, since

$$6 \cdot x^2 \cdot y^3 \cdot z = 6x^2y^3z$$

Of these, x^2, y^3, and z are literal factors, and 6 is the numerical factor, or *coefficient*. The factors 6, x^2, and y^3 can be factored further; *i.e.*, they are not *prime* factors. The prime factors of $6x^2y^3z$ are 2, 3, x, x, y, y, y, and z.

55. Common Monomial Factors. Since

$$3(x^2 + 2xy + z) \text{ is equivalent to } 3x^2 + 6xy + 3z$$

the expression $3x^2 + 6xy + 3z$ can be written in the *factored* form $3(x^2 + 2xy + z)$, with 3 and $(x^2 + 2xy + z)$ as its factors. The terms $3x^2$, $6xy$, and $3z$ are said to have the *common* numerical factor 3, which can be removed and placed outside the parentheses. Similarly, the terms $3x^2$, $2xy$, and $4xy^2$ have the common literal factor x; therefore the expression $3x^2 + 2xy + 4xy^2$ can be written in the factored form $x(3x + 2y + 4y^2)$.

One removes a factor common to the terms of an expression by dividing each term by the common factor and writing the result in parentheses, with the common factor outside.

Example: In factoring the expression $3ab^2c + 6a^2b + 9ac^2$, one observes that the three terms of this expression have the common literal factor a and the common numerical factor 3, or the common monomial factor $3a$. One therefore writes the expression in the *factored* form $3a(b^2c + 2ab + 3c^2)$.

The first step in factoring a multinomial is to remove any monomial factor common to its terms. One should take care that he removes the *greatest common factor* of the terms, *i.e.*, the greatest factor which they have in common, so that no common factor will remain inside the parentheses. Thus the expression $4xyz^2 + 6xy^2z + 2x^3y$ should be factored as $2xy(2z^2 + 3yz + x^2)$, not $2x(2yz^2 + 3y^2z + x^2y)$ or $xy(4z^2 + 6yz + 2x^2)$. In the last two cases, a common factor is left inside the parentheses, and the process of factoring has not been completed.

EXERCISES (55)

Factor the following expressions:

1. $3x + 15$ 2. $x^2 + 3x$ 3. $4x^2 + 2xy$ 4. $x^2y^2 + xy^2 - 2x^2y$
5. $ax^2 - ay^2 + a^2xy$ 6. $2m^2n + 4mn^2 - 6m^2n^3$ 7. $3x^2 - 6xy + 9y^2$
8. $3x^3 - 2x^2 + 7x$ 9. $3x^4 + 7x^3 - 12x^2$ 10. $2a^4b^3 + 6ab^4 + 10a^2b^2$

56. Factoring by Grouping. Consider the product

$$(x + 4)(y + 6)$$

which is equal to $xy + 6x + 4y + 24$. There is no factor that is common to all the terms of the expression $xy + 6x + 4y + 24$. Its factored form $(x + 4)(y + 6)$ can be obtained by *grouping* those of its terms which do contain a common factor. Observe that xy and $6x$ contain the common factor x, and that $4y$ and 24 contain the common factor 4.

Thus, $(xy + 6x) + (4y + 24)$

becomes $x(y + 6) + 4(y + 6)$

But these terms contain the common *binomial* factor $y + 6$; hence we may write

$$(x + 4)(y + 6)$$

which is the factored form of $xy + 6x + 4y + 24$.

Example 1. Factor $7x + xy + 7y + y^2$.

Solution:

$$7x + xy + 7y + y^2 = x(7 + y) + y(7 + y) = (x + y)(7 + y)$$

Note that it makes no difference how the terms are grouped:

$$7x + xy + 7y + y^2 = 7(x + y) + y(x + y) = (7 + y)(x + y)$$

Example 2. Factor $ac + ad + bc + bd$.

Solution:

$$ac + ad + bc + bd = a(c + d) + b(c + d) = (a + b)(c + d)$$

Example 3. Factor $a^3 - a^2 + 2a - 2$.

Solution:

$$a^3 - a^2 + 2a - 2 = a^2(a - 1) + 2(a - 1) = (a^2 + 2)(a - 1)$$

EXERCISES (56)

Factor by grouping:

1. $x^2 + xy + 3x + 3y$ 2. $3ax - bx - 3ay + by$

3. $4a - ma - 4b + mb$
4. $xy - 3x - 2y + 6$
5. $a^3 - b^2 - a^2 + ab^2$
6. $6x^3 - 4x^2 + 9x - 6$
7. $5a^3 + 5a^2 - a - 1$
8. $5ab - 5bc - 3a + 3c$
9. $12 - 4x + 3x^2 - x^3$
10. $ax + bx + ay + by - cx - cy$
11. $12x^2 - 8x^3 + 9 - 6x$
12. $a^3 + 1 + a^2 + a$

57. Factoring of Quadratic Trinomials. A *quadratic trinomial* is a trinomial of second degree. Consider the products

$$\begin{array}{rr} x+7 & \\ x+3 & \\ \hline x^2+7x & \\ 3x & +21 \\ \hline x^2+10x & +21 \end{array} \qquad \begin{array}{rr} x+7 & \\ x-3 & \\ \hline x^2+7x & \\ -3x & -21 \\ \hline x^2+4x & -21 \end{array} \qquad \begin{array}{rr} x-7 & \\ x+3 & \\ \hline x^2-7x & \\ 3x & -21 \\ \hline x^2-4x & -21 \end{array} \qquad \begin{array}{rr} x-7 & \\ x-3 & \\ \hline x^2-7x & \\ -3x & +21 \\ \hline x^2-10x & +21 \end{array}$$

In each case, the first term of the trinomial is the product of the first terms of the binomials, the last term of the trinomial is the product of the last terms of the binomials, and the second term of the trinomial is the algebraic sum of the *cross products*. The product of two binomials is a trinomial only when the corresponding terms of the two binomials are similar; *i.e.*, when they contain the same literal factors.

Example:

$$(x + 4)(x + 3) = x^2 + 7x + 12$$

but

$$(x + 4)(y + 3) = xy + 3x + 4y + 12$$

In order to factor a trinomial into two binomials, we must select first terms (for the binomials) that are the factors of the first term of the trinomial, and last terms that are the factors of the last term of the trinomial. Thus, in order to factor $x^2 + 4x - 21$, we arrange the following:

First Terms of the Binomials	Possible Last Terms of the Binomials			
x	-7	7	21	-21
and	and	and	and	and
x	3	-3	-1	1

We choose that arrangement of the last terms which makes the sum of the cross products equal to the middle term of the trinomial. The possible arrangements are

$$\begin{array}{cccc} x - 7 & x + 7 & x + 21 & x - 21 \\ \underline{x + 3} & \underline{x - 3} & \underline{x - 1} & \underline{x + 1} \\ x^2 - 4x - 21 & x^2 + 4x - 21 & x^2 + 20x - 21 & x^2 - 20x - 21 \end{array}$$

of which the second is the correct choice; therefore

$$x^2 + 4x - 21 = (x + 7)(x - 3)$$

It is more convenient to set up only this pattern:

$$\begin{array}{cc} x \quad 7 & x \quad 21 \\ \underline{x \quad 3} & \underline{x \quad\ 1} \end{array}$$

Recognizing that the last terms must be unlike in order to have -21 as a product, one can "juggle" the signs and compute the cross products mentally, in order to choose the proper arrangement. NOTE: Since $(-x - 7)(-x + 3) = (x + 7)(x - 3)$, it is not necessary to consider a choice of signs for the first terms.

Example 1. Factor $x^2 + 5x + 6$.

$$\begin{array}{cccc} x + 6 & x - 6 & \boxed{\begin{array}{c} x + 3 \\ x + 2 \end{array}} & x - 3 \\ \underline{x + 1} & \underline{x - 1} & & \underline{x - 2} \end{array}$$

Example 2. Factor $x^2 + 5x - 6$.

$$\begin{array}{cccc} \boxed{\begin{array}{c} x + 6 \\ x - 1 \end{array}} & \begin{array}{c} x - 6 \\ \underline{x + 1} \end{array} & \begin{array}{c} x + 3 \\ \underline{x - 2} \end{array} & \begin{array}{c} x - 3 \\ \underline{x + 2} \end{array} \end{array}$$

Example 3. Factor $2x^2 - x - 1$.

Set up $\begin{array}{cc} 2x & 1 \\ \underline{x} & \underline{1} \end{array}$ Choose $\begin{array}{c} 2x + 1 \\ \underline{x - 1} \end{array}$

When neither the first nor the last term of the trinomial has the coefficient 1, the number of possible choices is much greater.

Example 4. Factor $3x^2 - 7x - 6$.

Set up $\begin{array}{cccc} 3x \quad 6 & 3x \quad 1 & 3x \quad 3 & 3x \quad 2 \\ \underline{x \quad 1} & \underline{x \quad 6} & \underline{x \quad 2} & \underline{x \quad 3} \end{array}$

Choose $\begin{array}{c} 3x + 2 \\ \underline{x - 3} \\ 3x^2 - 7x - 6 \end{array}$

It should not be necessary to consider the third of the four arrangements shown, since $\begin{array}{cc} 3x & 3 \\ \underline{x} & \underline{2} \end{array}$ implies a common factor 3, whereas the original tri-

nomial should always be examined to see if a common monomial factor can be removed before attempting to factor into binomials.

Observe the following:

1. If the last term of the trinomial is positive, the last terms of the two binomial factors have the same sign as the middle term of the trinomial.

Examples:

$$x^2 + 5x + 6 = (x + 2)(x + 3), \qquad x^2 - 5x + 6 = (x - 2)(x - 3)$$

2. If the last term of the trinomial is negative, the last terms of the two binomial factors have *opposite* signs.

3. Not all quadratic trinonials are factorable. For example, $x^2 + 2x + 8$ is not factorable, since 8 has no factors whose sum is as small as 2.

EXERCISES (57)

Mentally determine the following products. *Write nothing* until the answer is complete in your mind; then write only the answer.

1. $\begin{array}{l} x + 6 \\ \underline{x - 1} \end{array}$ 2. $\begin{array}{l} x - 3 \\ \underline{x - 2} \end{array}$ 3. $\begin{array}{l} a + 4 \\ \underline{a - 5} \end{array}$

4. $\begin{array}{l} m + 3 \\ \underline{m - 6} \end{array}$ 5. $\begin{array}{r} 2x + 1 \\ \underline{x + 4} \end{array}$ 6. $\begin{array}{l} 3x + 4 \\ \underline{2x - 3} \end{array}$

7. $(x + 3)(x + 4)$ 8. $(x + 5)(x - 6)$ 9. $(x + 1)(x - 7)$
10. $(2x + 1)(x + 3)$ 11. $(3x + 2)(2x + 5)$ 12. $(4x - 3)(3x + 2)$

Mentally factor the following. *Write nothing* until the answer is complete in your mind, then write only the answer. Check the answer by mental multiplication of the factors.

13. $x^2 + 5x + 6$ 14. $x^2 + 8x + 7$ 15. $a^2 + 9a + 8$
16. $x^2 - 7x + 12$ 17. $x^2 + 3x - 28$ 18. $x^2 - 2x - 24$
19. $x^2 - 20x - 21$ 20. $x^2 - 12x - 85$ 21. $x^2 - 16x - 260$
22. $x^2 + 5xy + 6y^2$ 23. $x^2 - 7xy + 6y^2$ 24. $x^2 - xy - 6y^2$

Some of the following are very difficult to factor mentally; so write them out if you need to do so.

25. $2x^2 + 3x + 1$ 26. $3x^2 + 4x + 1$
27. $5x^2 + 25x + 30$ 28. $6x^2 + 7x + 1$
29. $6x^2 - 7x + 1$ 30. $6x^2 - x - 1$

31. $3x^2 + 10x + 3$
32. $3x^2 - 10x + 3$
33. $3a^2 + 8a - 3$
34. $3x^2 - x - 4$
35. $2x^2 - x - 28$
36. $2a^2 - 5a + 2$
37. $6x^2 - x - 7$
38. $x^2 - \frac{3}{2}x - 7$

39. $6 - 11a + 4a^2$ (It is not necessary to reverse the order.)

40. $4a^2 + 3ab - 27b^2$
41. $21a^2 - 22ab - 24b^2$

Did you do Exercise 27 "the hard way," or did you remove the common factor 5?

58. The Difference of Two Squares. Consider the product

$$\begin{array}{lll} x & + \, y & \\ x & - \, y & \\ x^2 & + \, xy & \\ & - \, xy & - \, y^2 \\ \hline x^2 & & - \, y^2 \end{array}$$

Thus $x^2 - y^2$ can be written in the factored form $(x + y)(x - y)$.

Example 1. $a^2 - b^2 = (a + b)(a - b)$
Example 2. $x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$
Example 3. $9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Example 4. $16a^2 - 9b^2 = (4a)^2 - (3b)^2 = (4a + 3b)(4a - 3b)$

EXERCISES (58)

Mentally obtain the following products, writing only the result:

1. $(3x + 5)(3x - 5)$
2. $(2a - 7)(2a + 7)$
3. $(6 - 5a)(6 + 5a)$
4. $(4x^2y + 3z^3)(4x^2y - 3z^3)$
5. $(x^3 + 1)(x^3 - 1)$
6. $(a^3 + b^3)(a^3 - b^3)$
7. $(2x^3 - 3y^3)(2x^3 + 3y^3)$
8. $(6x^5 - 5y^4)(6x^5 + 5y^4)$

Mentally factor the following, writing only the result:

9. $a^2 - 25$
10. $36 - x^2$
11. $9x^2 - 49y^2$
12. $81x^2 - 1$
13. $49x^2 - 25y^2$
14. $36a^2x^2 - 9b^2y^2$
15. $1 - 100x^2y^2$
16. $x^4y^6 - m^2n^4$
17. $.01x^2 - .01y^2$
18. $16x^{24} - 100m^{32}$
19. $1 - 64m^{12}n^{16}$
20. $(102)^2 - (98)^2$

59. Difference of Squared Binomials (Optional). The method of the preceding section can be applied to cases in which one (or both) of the squared terms is a binomial.

Example 1. $(x + 3)^2 - y^2 = [(x + 3) + y][(x + 3) - y]$
$= (x + 3 + y)(x + 3 - y)$

Example 2. $(x + y)^2 - (a + b)^2$
$$= [(x + y) + (a + b)][(x + y) - (a + b)]$$

Example 3. $x^2 + 6x + 9 - y^2 = (x + 3)^2 - y^2$
$$= [(x + 3) + y][(x + 3) - y]$$

Example 4. $4x^2 + 12xy + 9y^2 - a^2 + 4ab - 4b^2$
$$= (4x^2 + 12xy + 9y^2) - (a^2 - 4ab + 4b^2)$$
$$= (2x + 3y)^2 - (a - 2b)^2$$
$$= [(2x + 3y) + (a - 2b)][(2x + 3y) - (a - 2b)]$$

EXERCISES (59)

Factor the following:

1. $(x + y)^2 - z^2$
2. $a^2 - (b + c)^2$
3. $x^2 - (y - z)^2$
4. $4x^2 - (2y + z)^2$
5. $(a + b)^2 - (c - d)^2$
6. $(3m - 2n)^2 - 1$
7. $x^2 + 2xy + y^2 - a^2 - 2ab - b^2$
8. $4x^2 - 4xy + y^2 - x^2 - 2xy - y^2$ NOTE: This can be factored as

$$(2x - y)^2 - (x + y)^2 = [(2x - y) + (x + y)][(2x - y) - (x + y)]$$
$$= (3x)(x - 2y),$$

but much time is saved if similar terms are combined at the start.

60. Solution of Equations by Factoring. At the beginning of the chapter it was shown that an algebraic expression is equal to zero if any one of its factors is zero. This fact leads to the following rule:

> ***In an equation, if $x - a$ is a factor of the expression that is equated to zero, then $x - a = 0$ will yield a solution of the equation.***

Example 1. Solve: $x^2 - 2x - 3 = 0$
Solution: Factoring, $(x - 3)(x + 1) = 0$

Setting each factor equal to zero,

$$x - 3 = 0 \qquad x + 1 = 0$$
$$x = 3 \qquad x = -1$$

Thus $x = 3$ and $x = -1$ are the solutions.
Checking, $(3)^2 - 2(3) - 3 \equiv 0$, checks
and $(-1)^2 - 2(-1) - 3 \equiv 0$, checks.

When x is one of the factors of the expression equated to zero, $x = 0$ is a solution:

Example 2. Solve: $x^3 - 4x^2 + 3x = 0$

Factoring, $x(x - 3)(x - 1) = 0$

and the roots are $x = 0$, $x = 3$, and $x = 1$

Check these roots by substitution.

EXERCISES (60)

Solve by factoring, and check:

1. $x^2 - 5x + 6 = 0$
2. $x^2 - 8x + 7 = 0$
3. $x^2 + 7x + 12 = 0$
4. $x^2 - 7x = 0$
5. $x^2 - ax = 0$
6. $x^2 - 16x - 260 = 0$
7. $x^3 - 5x^2 + 6x = 0$
8. $2x^2 + 3x + 1 = 0$
9. $6x^2 - x - 1 = 0$
10. $2a^2 - a - 28 = 0$
11. $x^2 - 36 = 0$
12. $x^3 - 25x = 0$
13. $x^2 - 4ax + 4a^2 = 0$
14. $(x - 3)^2 - 9 = 0$
15. $16x^2 - 25a^2 = 0$

61. The Factor Theorem (Optional). One of the most useful methods of factoring is based on the following theorem:

> ***The Factor Theorem***
>
> ***If an expression in x becomes equal to zero when a is substituted for x, then x − a is a factor of the expression.***

The use of the theorem will be illustrated:

Example 1. Factor the expression $x^3 - y^3$.

Solution: Observe that if y is substituted for x, the expression above becomes $y^3 - y^3$, which equals zero. Then $x - y$ is a factor of the expression, and we can obtain the remaining factor by division:

$$\begin{array}{r|l} x^3 \qquad\qquad\qquad - y^3 & x - y \\ \hline x^3 - x^2y \qquad\qquad\quad & x^2 + xy + y^2 \\ x^2y \qquad\qquad & \\ x^2y - xy^2 \qquad & \\ + xy^2 - y^3 & \\ xy^2 - y^3 & \end{array}$$

Thus, $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Example 2. Factor $x^5 + 32$.

Solution: Observe that if -2 is substituted for x, the expression $x^5 + 32$ becomes $-32 + 32$, or zero. Thus $x - (-2)$, or $x + 2$, is a factor. Dividing,

$$
\begin{array}{r|l}
x^5 \qquad\qquad\qquad + 32 & x + 2 \\
\hline
x^5 + 2x^4 \qquad\qquad\qquad & x^4 - 2x^3 + 4x^2 - 8x + 16 \\
- 2x^4 \qquad\qquad\quad & \\
- 2x^4 - 4x^3 \qquad\quad & \\
+ 4x^3 \qquad\quad & \\
4x^3 + 8x^2 \qquad & \\
- 8x^2 \qquad & \\
- 8x^2 - 16x \quad & \\
+ 16x + 32 & \\
+ 16x + 32 &
\end{array}
$$

Then $\qquad x^5 + 32 = (x + 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$

In seeking values of x that reduce the multinomial to zero, test only those numbers which are *factors of the term of lowest degree.*

Example 3. Factor $x^3 - x - 6$.

Solution: Here one should test only 3, -3, 2, -2, 6, -6, 1, -1.
Trying 3, we obtain $27 - 3 - 6 = 0$, does not check.
Trying -3, we obtain $-27 + 3 - 6 = 0$, does not check.
Trying 2, we obtain $8 - 2 - 6 = 0$, checks; hence $x - 2$ is a factor.
Dividing,

$$
\begin{array}{r|l}
x^3 \qquad\qquad - x - 6 & x - 2 \\
\hline
x^3 - 2x^2 \qquad\qquad & x^2 + 2x + 3 \\
+ 2x^2 - x \qquad & \\
2x^2 - 4x \qquad & \\
+ 3x - 6 & \\
3x - 6 &
\end{array}
$$

Then $\qquad x^3 - x - 6 = (x - 2)(x^2 + 2x + 3)$

In general, of course, it is necessary to determine *by trial* the proper number to substitute for x. However, it is best to study the following cases:

Law I. $x^n + y^n$, if n is odd, is divisible by $x + y$.

Example: $\qquad x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Law II. $x^n - y^n$, if n is odd, is divisible by $x - y$.

Example: $\qquad x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Law III. $x^n - y^n$, if n is even, can be written as the difference of two squares.

Example: $x^6 - y^6 = (x^3)^2 - (y^3)^2 = (x^3 - y^3)(x^3 + y^3)$

Law IV. $x^n + y^n$, if n contains an *odd* number as a factor, can be simplified to the form of case I.

Example: $x^{10} + y^{10} = (x^2)^5 + (y^2)^5$ is divisible by $x^2 + y^2$.

Law V. $x^n + y^n$, if n contains no odd factor, is not factorable.
Example: $x^8 + y^8$ is not factorable.

Example 4. Factor $x^6 - y^6$.
Solution: $x^6 - y^6 = (x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3)$

Since $x^3 + y^3$ is divisible by $x + y$, we divide:

$$
\begin{array}{r|l}
x^3 \qquad\qquad\qquad + y^3 & x + y \\ \hline
x^3 + x^2y \qquad\qquad\qquad & x^2 - xy + y^2 \\
\hline
- x^2y \qquad\qquad & \\
- x^2y - xy^2 \qquad & \\
\hline
+ xy^2 + y^3 & \\
xy^2 + y^3 & \\
\hline
\end{array}
$$

Since $x^3 - y^3$ is divisible by $x - y$, we divide:

$$
\begin{array}{r|l}
x^3 \qquad\qquad\qquad - y^3 & x - y \\ \hline
x^3 - x^2y \qquad\qquad\qquad & x^2 + xy + y^2 \\
\hline
+ x^2y \qquad\qquad & \\
x^2y - xy^2 \qquad & \\
\hline
+ xy^2 - y^3 & \\
xy^2 - y^3 & \\
\hline
\end{array}
$$

Then $\quad x^6 - y^6 = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$

EXERCISES (61)

Use the factor theorem to show that

1. $x - 1$ is a factor of $x^2 - 7x + 6$
2. $x - 2$ is a factor of $x^2 + 3x - 10$
3. $x + 3$ is a factor of $x^3 + x^2 - 2x + 12$
4. $m + 2$ is a factor of $2m^3 - 4m^2 + m + 34$
5. $x + 2$ and $x - 1$ are factors of $x^3 + 7x^2 + 4x - 12$
6. $x - 3$ and $x + 1$ are factors of $x^4 - 7x^2 - 6x$

Write as a product of two factors:

7. $x^3 + x + 10$
8. $x^3 - 3x^2 - 16$
9. $z^3 - 2z^2 + z - 12$
10. $m^3 + 2m^2 - 7m - 8$

11. $x^9 + 1$
12. $x^3 - 27$
13. $x^{12} + y^{12}$
14. $a^8 - b^8$

Write as a product of three factors:

15. $x^3 - x^2 - 17x - 15$
16. $n^3 + 6n^2 + 11n + 6$
17. $x^4 + x^3 - x^2 - 19x - 42$
18. $r^4 - 3r^3 + 6r - 4$
19. $r^{12} - s^{12}$
20. $x^9 - y^9$

Factor completely:

21. $b^3 - 3b - 2$
22. $x^3 + 3x^2 - 4$
23. $x^4 - 7x^3 + x^2 + 3x - 70$
24. $x^4 + 7x^3 + 10x^2 - x - 5$
25. $a^3 + 3a^2b + 3ab^2 + b^3$
26. $m^3 - 7m^2n + 11mn^2 - 5n^3$
27. $x^4 - y^4$
28. $x^6 + y^6$
29. $x^{15} - 1$
30. $x^8 - y^8$

62. General Method of Factoring. The general procedure for factoring any expression is as follows:

1. Remove any common monomial factor; then
2. If the expression has two terms, try to factor it as the difference of two squares.
3. If the expression is a quadratic trinomial, try to factor it into two binomials.
4. If it has four or more terms, try to factor it by grouping.
5. When (1) to (4) do not apply, try the factor theorem.

EXERCISES (62)

Following the general procedure just outlined, factor the following:

1. $3x^3 + 6x^2 + 3x$
2. $9a^2b^3 + 3a^2b^2$
3. $6x^4y^5 + 2x^3y^4 + 3x^7y^6$
4. $4a^2 - 9b^2$
5. $m^2 - mx + my - xy$
6. $x^2 - xy + xz - yz$
7. $4a^3 + 8a^2b - 9ab^2 - 18b^3$
8. $x^2 + 2x - 63$
9. $a^2 + 4a - 77$
10. $3x^2 + 5x - 12$
11. $(a + b)^2 - a^2$
12. $x^3 + x + 3x^2 - 1$
13. $(x^{10} - y^{10})$
14. $15a^2 - 8a - 12$
15. $x^2 - 3x - xy + 3y$
16. $x^3 - 3x^2 + 7x - 21$
17. $x^2y^2 - x^2 - y^2 + 1$
18. $(x - 4y)^2 - 9z^2$
19. $m^4 + 5m^3 + 5m^2 - 5m - 6$
20. $x^4 - 4x^3 + 4x^2$
21. $n^3 - 7n + 6$
22. $16a^2x^2 - 49a^2$
23. $a^4 - 2a^3 + a^2 - 4$
24. $4x^8 - 4y^8$
25. $2x^4 + 6x^3 - 8x^2 - 24x$
26. $x^6 - 1$

27. $16x^5y^4 - x$

28. $a^4 - a^3 + 4a - 16$

29. $x^2 - y^2 + 2y - 2x$

30. $x^4 - 1$

REVIEW QUESTIONS

1. What are the two most important uses of factoring?

2. Describe the general procedure for factoring.

3. Describe two ways in which you might prove that $x = 2$ is a solution of the equation $x^2 - 5x + 6 = 0$.

4. Describe two ways of solving $x^2 - 5x + 6 = 0$.

CHAPTER 8

FRACTIONS

An algebraic fraction is the indicated quotient of two algebraic numbers or expressions. Such fractions occur repeatedly in the equations encountered in all branches of applied science and engineering; hence it is important to be able to handle them with facility.

63. Reduction to Lowest Terms. A fraction is said to be in its lowest (simplest) terms when its numerator and denominator contain no common factor. The following rule is used in reducing a fraction to its lowest terms:

> ***The value of a fraction is not changed by multiplying or dividing both its numerator and its denominator by the same quantity (not zero).***

According to this rule, any factor that appears in *both* numerator and denominator can be removed by dividing both numerator and denominator by the common factor. When all common factors have been removed, the fraction is in its lowest terms.

Example 1. Reduce to lowest terms $\frac{12a^3b^5}{20a^2b^3c}$.

Solution: The numerator and denominator contain the common factors 4, a^2, and b^3. Dividing both numerator and denominator by each of these, or by their product $4a^2b^3$, one obtains $\frac{3ab^2}{5c}$, which is in its lowest terms.

Example 2. Simplify $\frac{2x^2 - xy - 6y^2}{3x - 6y}$.

Note: It is usually best to write both the numerator and denominator in factored form before performing any division.

Solution: $\frac{2x^2 - xy - 6y^2}{3x - 6y} = \frac{(2x + 3y)(x - 2y)}{3(x - 2y)} = \frac{2x + 3y}{3}$

in lowest terms.

Example 3. Simplify $\dfrac{9x^2 - 4y^2}{6xy + 4y^2}$

Solution: $\dfrac{9x^2 - 4y^2}{6xy + 4y^2} = \dfrac{(3x + 2y)(3x - 2y)}{2y(3x + 2y)} = \dfrac{3x - 2y}{2y}$

in lowest terms.

EXERCISES (63)

Reduce to lowest terms:

1. $\frac{4}{12}$ 2. $\frac{20}{25}$ 3. $\frac{63}{36}$ 4. $\frac{187}{148}$

5. $\dfrac{21a^3b^2}{28ab^5}$ 6. $\dfrac{56a^3b^3c^2}{32b^3c^7}$ 7. $\dfrac{51m^6n^4}{68m^2n}$

8. $\dfrac{x^2 - y^2}{x + y}$ 9. $\dfrac{a - b}{a^2 - b^2}$ 10. $\dfrac{m^2 + 6mn + 9n^2}{m + 3n}$

11. $\dfrac{x^2 - 2x + 1}{x^2 - 1}$ 12. $\dfrac{ax^2 + abx}{cx^2 + cbx}$ 13. $\dfrac{x^2 + x - 56}{x^2 + 6x - 16}$

14. $\dfrac{x^2 - 2x - 63}{x^2 - 11x + 18}$ 15. $\dfrac{a^2 - 7a + 10}{a^2 - 8a + 15}$ 16. $\dfrac{a^2 - b^2}{(a - b)^2}$

17. $\dfrac{a^2 - 2ab + b^2}{a^2 - 3ab + 2b^2}$ 18. $\dfrac{m^2 + mn}{m^2 - n^2}$ 19. $\dfrac{18x^3y^2 + 12x^2y^3}{18x^3y^2 + 27x^4y}$

20. $\dfrac{2a^2 - a - 3}{2a^2 - 5a + 3}$ 21. $\dfrac{3m^2 - 11m + 10}{25 - 9m^2}$ 22. $\dfrac{16y^2 - x^2y^2}{x^2y + 9xy + 20y}$

23. $\dfrac{m^2n^2 - 1}{m^2n^2 - mn}$ 24. $\dfrac{a^2x^3 + 5a^2x^2 + 4a^2x}{ax^4 - 3ax^3 - 4ax^2}$

25. $\dfrac{x^5y^4 - x^3y^2}{x^3y^4 + x^2y^3}$ 26. $\dfrac{ax - ay + bx - by}{ax + ay + bx + by}$

27. $\dfrac{a^3 - 3a^2 + a - 3}{a^2 - 9}$ 28. $\dfrac{2x^3 - 2x^2 - 3x + 3}{2x^3 + 2x^2 - 3x - 3}$

64. Multiplication and Division of Fractions. In multiplying fractions, the following rule is used:

> ***In multiplying two or more fractions, multiply the numerators to obtain the numerator of the product; then multiply the denominators to obtain the denominator of the product.***

Example 1. Multiply $\dfrac{3x}{2y} \cdot \dfrac{8y^2}{9x^2}$, and simplify the result.

Solution: $\dfrac{3x}{2y} \cdot \dfrac{8y^2}{9x^2} = \dfrac{24xy^2}{18x^2y} = \dfrac{4y}{3x}$

NOTE: The numerator and denominator of each fraction should be written in factored form *before* the multiplication is performed.

Example 2. Multiply $\dfrac{x^2 - 5x + 6}{x^2 - 9} \cdot \dfrac{x^2 + 4x + 3}{x^2 + 8x + 7}$.

Solution: If one proceeds to multiply these fractions before factoring, the numerator and denominator of the product will be very cumbersome, and the product will not be easy to simplify. If the fractions are first written in factored form, however, the multiplication can be performed without difficulty, *viz.*,

$$\frac{x^2 - 5x + 6}{x^2 - 9} \cdot \frac{x^2 + 4x + 3}{x^2 + 8x + 7} = \frac{(x-2)(x-3)}{(x+3)(x-3)} \cdot \frac{(x+3)(x+1)}{(x+7)(x+1)}$$
$$= \frac{x-2}{x+3} \cdot \frac{x+3}{x+7} = \frac{(x-2)(x+3)}{(x+7)(x+3)} = \frac{x-2}{x+7}$$

Not all the above steps are necessary. With the product in the form $\dfrac{(x-2)\cancel{(x-3)}}{\cancel{(x+3)}\cancel{(x-3)}} \cdot \dfrac{\cancel{(x+3)}\cancel{(x+1)}}{(x+7)\cancel{(x+1)}}$, one can obtain the result $\dfrac{x-2}{x+7}$ by *canceling* the factors that are to be divided out, as shown.

Example 3. Simplify $\dfrac{x^2 - x}{x^2 + 2x} \cdot \dfrac{x^2 - 4}{x^2 - 1} \cdot \dfrac{x^2 + x}{x^2 - 2x}$.

Solution: $\dfrac{x^2 - x}{x^2 + 2x} \cdot \dfrac{x^2 - 4}{x^2 - 1} \cdot \dfrac{x^2 + x}{x^2 - 2x}$

$$= \frac{\cancel{x}\cancel{(x-1)}}{\cancel{x}\cancel{(x+2)}} \cdot \frac{\cancel{(x+2)}\cancel{(x-2)}}{\cancel{(x+1)}\cancel{(x-1)}} \cdot \frac{\cancel{x}\cancel{(x+1)}}{\cancel{x}\cancel{(x-2)}} = 1$$

Note that if everything cancels out the result is unity, not zero; since each cancellation leaves the factor 1 in place of the factor canceled.

> ***To divide one fraction by another, invert the divisor and proceed as in multiplying fractions.***

Example 4. Divide $\frac{2}{9}$ by $\frac{3}{7}$.

Solution: $\frac{2}{9} \div \frac{3}{7} = \frac{2}{9} \cdot \frac{7}{3} = \frac{14}{27}$

Example 5. Divide $\dfrac{24ab^4}{10b^3}$ by $\dfrac{12a^2b}{18a^3}$.

Solution: $\dfrac{24ab^4}{10b^3} \div \dfrac{12a^2b}{18a^3} = \dfrac{24ab^4}{10b^3} \cdot \dfrac{18a^3}{12a^2b} = \dfrac{18a^2}{5}$

Example 6. Divide $\dfrac{a^2 + a - 20}{a^2 - 4a}$ by $\dfrac{a + 5}{a^2 - 2a}$.

Solution:

$$\frac{a^2 + a - 20}{a^2 - 4a} \div \frac{a + 5}{a^2 - 2a} = \frac{(a + 5)(a - 4)}{a(a - 4)} \cdot \frac{a(a - 2)}{a + 5} = a - 2$$

(Note that the inversion and factoring were done in a single step.)

The sign of a fraction is the sign that precedes the fraction bar. If omitted, it is assumed to be positive. One may change the sign of a fraction without changing the value of the fraction, if at the same time he changes the sign of a factor in either the numerator or denominator.

Example 7. $\frac{x - 1}{2(x + 1)} = -\frac{x - 1}{-2(x + 1)} = \frac{1 - x}{-2(x + 1)}$

Example 8.

$$\frac{a^2 - x^2}{x + a} = -\frac{x^2 - a^2}{x + a} = -\frac{(x + a)(x - a)}{x + a} = a - x$$

EXERCISES (64)

Perform the operations indicated:

1. $\frac{9}{10} \cdot \frac{5}{27}$

2. $\frac{65}{139} \cdot \frac{34}{52}$

3. $\frac{25a^2}{7ab^3} \cdot \frac{28b^2}{15a^2}$

4. $\frac{63xy^3z^3}{26a^3bc^2} \cdot \frac{13a^2bc^4}{14x^2y^3z}$

5. $\frac{21}{10} \div \frac{3}{4}$

6. $\frac{a^4}{4b^3} \div \frac{5a}{2b^2}$

7. $\frac{4x^2 - y^2}{x^2 - y^2} \cdot \left(-\frac{x + y}{2x + y}\right)$

8. $\frac{x^2 - y^2}{3x} \cdot \frac{x}{x + y}$

9. $\frac{3x^2 - 9x}{4x - 12} \cdot \frac{2x^3 + x^2}{6x^2 + 3x}$

10. $\frac{2a^2 + 9a + 4}{a^2 + 4a} \cdot \frac{2a^2 - 8}{2a^2 + 5a + 2}$

11. $\frac{x - y}{x + y} \div \frac{x - y}{-1}$

12. $\frac{x^2 - 4}{x^2 - 49} \div \frac{x + 2}{x + 7}$

13. $\frac{18a^3 + 36a^2}{21a^3 - 7a} \div \frac{a^2 + 2a}{2a - 1}$

14. $\frac{m^3n - m^2n^2}{m^3n^3 + m^2n^4} \div \frac{m^2n^3 - mn^4}{m^4n^3 + m^3n^4}$

15. $\frac{x^2 - 2ax + a^2}{2a^2 - 4ax} \cdot \frac{2x^2 - ax}{a^2 - x^2}$

16. $\frac{2r^2 - 3rs + s^2}{r^2 - s^2} \cdot \frac{r - s}{2r - s}$

17. $\frac{x^2y - y^3}{x^2y - xy^2} \cdot \frac{x^2 - xy}{x^4 - y^4}$

18. $\frac{21x - 35}{6x + 3} \cdot \frac{12x + 6}{2x - 18} \cdot \frac{4x - 36}{3x - 5}$

19. $\frac{4a^2 - 4}{3 + 2a} \cdot \frac{4a^2 - 9}{2a - 2} \cdot \frac{8a - 14}{2a^2 - a - 3}$

20. $\dfrac{m+1}{m^2+5m} \cdot \dfrac{m^2-25}{m^2-m-20} \div \dfrac{m^2-m-2}{m^2+2m-8}$

21. $\dfrac{2x+2}{x+5} \cdot \dfrac{4x-12}{3x+3} \div \left(-\dfrac{3-x}{3x+15}\right)$

22. $\dfrac{x^2+6x+9}{x^2-x-12} \cdot \dfrac{x^2-3x+2}{x^2+x-6} \div \dfrac{x^2+x-2}{x^2-6x+8}$

23. $\dfrac{x^2-xy}{x^4} \cdot \dfrac{x+y}{y^2} \div \dfrac{x^2-y^2}{x^2y^2}$

24. $\dfrac{x^2+y^2}{x^4-y^4} \cdot \dfrac{x^2-xy}{2x^2-3xy} \cdot \dfrac{4x^2-8xy+3y^2}{2x-y}$

25. $\dfrac{a^2-a}{a^2+2a} \div \dfrac{a^2-2a}{a^2+a} \cdot \dfrac{a^2-1}{a^2-4}$ (NOTE: The first fraction is to be divided by the product of the other two.)

26. $\dfrac{x^2+3x}{x+1} \cdot \dfrac{x^2-x-2}{x^2-9} \cdot \dfrac{x^2-x-20}{x^2+2x-8}$

27. $\dfrac{12-2x}{4x+x^2} \div \dfrac{2x^2-2}{8+2x} \cdot \dfrac{12+2x}{12-12x}$

28. $\dfrac{x^2+5x+6}{x^2+2x-35} \cdot \dfrac{x^2-3x-10}{x^2+9x+14} \cdot \dfrac{x^2+4x-21}{2x^2-8}$

65. Addition and Subtraction of Fractions. In algebra, as in arithmetic, one can add or subtract fractions only if their denominators are equal. In order to add or subtract fractions, therefore, it is necessary first to rearrange them (without changing their values) so that their denominators are equal, after which their numerators can be added or subtracted. The procedure is as follows:

> ***In order to add or subtract fractions:***
>
> ***1. Express them in terms of their lowest common denominator.***
>
> ***2. Add or subtract their numerators, retaining the common denominator.***

The lowest common denominator (L.C.D.) of a group of fractions is determined as follows: Reduce the fractions to lowest terms, leaving the denominators in factored form. List all the factors that appear in any of the denominators. (If a factor appears in a power higher than the first, list its highest power.) The lowest common denominator is the product of these factors.

Example 1. Find the L.C.D. of $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{35}$.

Solution: In factored form, the fractions are $\frac{1}{2}$, $\frac{3}{2^2}$, $\frac{5}{3 \times 2}$, $\frac{7}{7 \times 5}$.
Note that $\frac{7}{35}$, in lowest terms, is $\frac{1}{5}$. The L.C.D. must contain the factors 2^2, 3, 5; hence it is $2^2 \times 3 \times 5 = 60$.

Example 2. Find the L.C.D. of $\frac{2}{5xy}$, $\frac{3}{x^2}$, $\frac{2z}{6yz^2}$.

Solution: In lowest terms, the fractions are $\frac{2}{5xy}$, $\frac{3}{x^2}$, $\frac{1}{3yz}$. The L.C.D. is the product of 5, x^2, 3, y, and z, or $15x^2yz$.

In order to express fractions in terms of their L.C.D., one multiplies both numerator and denominator of each fraction by such factors as are needed to change the denominator of the fraction to the L.C.D. In the preceding example,

$$\frac{2}{5xy} = \frac{2}{5xy} \cdot \frac{3xz}{3xz} = \frac{6xz}{15x^2yz}$$
$$\frac{3}{x^2} = \frac{3}{x^2} \cdot \frac{15yz}{15yz} = \frac{45yz}{15x^2yz}$$
$$\frac{1}{3yz} = \frac{1}{3yz} \cdot \frac{5x^2}{5x^2} = \frac{5x^2}{15x^2yz}$$

and each fraction is now expressed in terms of the L.C.D. The fractions can now be added *viz.*,

$$\frac{2}{5xy} + \frac{3}{x^2} + \frac{2z}{6yz^2} = \frac{6xz}{15x^2yz} + \frac{45yz}{15x^2yz} + \frac{5x^2}{15x^2yz} = \frac{6xz + 45yz + 5x^2}{15x^2yz}$$

The method can be simplified somewhat by dividing each denominator into the L.C.D. to find the number by which the numerator must be multiplied. The intermediate steps can then be omitted, as follows:

Example 3. Add $\frac{3}{2x} + \frac{2}{y} + \frac{5}{z}$

Solution: The L.C.D. is $2xyz$, and the sum is

$$\frac{3 \cdot \frac{2xyz}{2x} + 2 \cdot \frac{2xyz}{y} + 5 \cdot \frac{2xyz}{z}}{2xyz} = \frac{3yz + 4xz + 10xy}{2xyz}$$

Example 4. Simplify $\dfrac{4}{x^2-1} + \dfrac{x}{1-x} - \dfrac{6}{x+1}$.

Solution: Note that $\qquad \dfrac{x}{1-x} = -\dfrac{x}{x-1}$

Then $\dfrac{4}{x^2-1} + \dfrac{x}{1-x} - \dfrac{6}{x+1} = \dfrac{4}{(x+1)(x-1)} - \dfrac{x}{x-1} - \dfrac{6}{x+1}$

The L.C.D. is $(x+1)(x-1)$; hence the sum is

$$\frac{4 - x(x+1) - 6(x-1)}{(x+1)(x-1)} = \frac{4 - x^2 - x - 6x + 6}{(x+1)(x-1)} = \frac{-x^2 - 7x + 10}{(x+1)(x-1)}$$
$$= \frac{x^2 + 7x - 10}{(x+1)(1-x)}$$

EXERCISES (65)

Combine and simplify:

1. $\dfrac{a}{2} + \dfrac{b}{3} + \dfrac{c}{5}$
2. $\dfrac{5x}{4} - \dfrac{2x}{3} + \dfrac{7x}{6}$
3. $\dfrac{5}{4x^3} - \dfrac{3}{4x} + \dfrac{7}{5x^2}$
4. $\dfrac{x^2-y^2}{xy} - \dfrac{x+y}{x} - \dfrac{x-y}{y}$
5. $\dfrac{a-b}{ab} + \dfrac{a-c}{ac} + \dfrac{2}{a}$
6. $\dfrac{2a-1}{a} + \dfrac{a+b}{ab} - \dfrac{b+1}{b}$
7. $\dfrac{5}{2x-4y} - \dfrac{3}{3x-6y}$
8. $\dfrac{x}{x+y} - \dfrac{y}{x-y}$
9. $\dfrac{1}{2a-1} - \dfrac{1}{2a-2} - \dfrac{1}{1-a}$
10. $\dfrac{2a-b}{a+2b} + \dfrac{3a-b}{a-2b}$
11. $\dfrac{x-7}{x-5} - \dfrac{x-4}{x-2}$
12. $\dfrac{s}{s+3} - \dfrac{2s}{s+2} + \dfrac{s}{s+1}$
13. $\dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{1}{x-3}$
14. $\dfrac{x}{x-y} - \dfrac{2xy}{x^2-y^2}$
15. $\dfrac{m-1}{2m+1} - \dfrac{2m+5}{2m-4} + \dfrac{2m-1}{4m+2}$
16. $\dfrac{1}{m+n} + \dfrac{1}{m-n} - \dfrac{2m}{m^2-n^2}$
17. $\dfrac{3x}{1-4x^2} + \dfrac{x}{1-2x} + \dfrac{6}{2x+1}$
18. $\dfrac{7}{4x-x^2} - \dfrac{2}{x^2-16}$
19. $\dfrac{3}{1+m} - \dfrac{m}{m^2-1} + \dfrac{2}{m^2+1}$
20. $\dfrac{1}{x^2-4x+3} + \dfrac{1}{x^2-5x+6}$
21. $\dfrac{6}{2x^2+x-1} - \dfrac{10}{2x^2+3x-2}$
22. $\dfrac{3}{a^2-5ab+6b^2} + \dfrac{12}{a^2-4b^2}$
23. $\dfrac{1}{x^2+3xy+2y^2} + \dfrac{1}{x^2-y^2} - \dfrac{1}{x^2+xy-2y^2}$
24. $\dfrac{2}{x^2-4x+3} + \dfrac{4}{x^2-2x-3} - \dfrac{2}{x^2-1}$

25. $2 + \dfrac{1}{x^2 - 1} + \dfrac{2}{x + 1}$

26. $1 + \dfrac{2a + b}{a - b} - \dfrac{6ab}{a^2 - b^2}$

27. $\dfrac{1}{2 - 3x + x^2} - \dfrac{5}{x^2 - 5x + 6}$

28. $\dfrac{5}{x^2 - x + 6} + \dfrac{7}{10 - 3x - x^2}$

29. $\dfrac{a^2 - a + 1}{a - 1} - \dfrac{a^2 + a + 1}{a + 1}$

30. $\dfrac{4x + 8}{2x^2 - x - 10} - \dfrac{10}{2x^2 - 5x}$

66. Complex Fractions. A *complex fraction* is one that has a fraction in either the numerator or the denominator, or in both. Thus, $\dfrac{\frac{a+b}{c}}{27}$, $\dfrac{\frac{abc}{a}}{a-b}$, and $\dfrac{\frac{x}{y} - \frac{y}{x}}{\frac{1}{y} + \frac{1}{x}}$ are complex fractions.

In order to simplify a complex fraction, reduce the numerator and denominator to single fractions; then invert the denominator and multiply by the numerator.

Example 1. Simplify $\dfrac{\frac{5}{12} - \frac{1}{4}}{\frac{2}{3} - \frac{1}{2}}$.

Solution:

$$\frac{\frac{5}{12} - \frac{1}{4}}{\frac{2}{3} - \frac{1}{2}} = \frac{\frac{5-3}{12}}{\frac{4-3}{6}} = \frac{\frac{2}{12}}{\frac{1}{6}} = \frac{2}{12} \cdot \frac{6}{1} = 1$$

Example 2. Simplify $\dfrac{\frac{9}{a^2} - \frac{4}{b^2}}{\frac{2}{b} + \frac{3}{a}}$.

Solution:

$$\frac{\frac{9}{a^2} - \frac{4}{b^2}}{\frac{2}{b} + \frac{3}{a}} = \frac{\frac{9b^2 - 4a^2}{a^2b^2}}{\frac{2a + 3b}{ab}} = \frac{9b^2 - 4a^2}{a^2b^2} \cdot \frac{ab}{2a + 3b}$$

$$= \frac{(3b - 2a)(3b + 2a)}{ab} \cdot \frac{1}{2a + 3b} = \frac{3b - 2a}{ab}$$

Example 3. Simplify $\dfrac{x + \frac{1}{x}}{x - \frac{1}{x}}$.

Solution: $\dfrac{x+\dfrac{1}{x}}{x-\dfrac{1}{x}} = \dfrac{\dfrac{x^2+1}{x}}{\dfrac{x^2-1}{x}} = \dfrac{x^2+1}{x} \cdot \dfrac{x}{x^2-1} = \dfrac{x^2+1}{x^2-1}$

Note that the solution would have been made simpler by multiplying both numerator and denominator by x; *viz.*,

$$\frac{x+\dfrac{1}{x}}{x-\dfrac{1}{x}} = \frac{x^2+1}{x^2-1}$$

Example 4. Simplify $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{x+y}$.

Solution: $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{x+y} = \dfrac{\dfrac{y+x}{xy}}{\dfrac{x+y}{1}} = \dfrac{y+x}{xy} \cdot \dfrac{1}{x+y} = \dfrac{1}{xy}$

Note that the denominator $x+y$, though not a fraction, is written in the form $\dfrac{y+x}{1}$ in preparation for its inversion.

EXERCISES (66)

Simplify:

1. $\dfrac{\dfrac{a}{b}}{c}$ 2. $\dfrac{a}{\dfrac{b}{c}}$ 3. $\dfrac{\dfrac{3x^3y^2}{4xy}}{\dfrac{9x^3y}{2xy^2}}$ 4. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x}+\dfrac{1}{y}}$ 5. $\dfrac{\dfrac{m}{3}+\dfrac{n}{4}}{\dfrac{m}{4}+\dfrac{n}{3}}$ 6. $\dfrac{\dfrac{1}{x}+1}{1+x}$

7. $\dfrac{a-\dfrac{1}{a}}{a-1}$ 8. $\dfrac{a+1}{1+\dfrac{1}{a}}$ 9. $\dfrac{\dfrac{x}{y}+x}{\dfrac{1}{y}+1}$ 10. $\dfrac{1-\dfrac{m}{n}}{\dfrac{n}{m}-1}$

11. $\dfrac{3+\dfrac{5x}{6y}}{x+\dfrac{18y}{5}}$ 12. $\dfrac{x-3+\dfrac{2}{x}}{x-1-\dfrac{2}{x}}$ 13. $\dfrac{x-5+\dfrac{6}{x}}{x-2-\dfrac{3}{x}}$ 14. $\dfrac{\dfrac{x^2}{y^2}-1}{\dfrac{x}{y}-1}$

15. $\dfrac{\dfrac{m^2}{n^2}-1}{\dfrac{m^2}{n^2}-\dfrac{2m}{n}+1}$ 16. $\dfrac{a}{\dfrac{a-1}{a+1}-1}$ 17. $\dfrac{\dfrac{1}{x+y}-\dfrac{1}{x-y}}{\dfrac{3}{x^2-y^2}}$

18. $\dfrac{\dfrac{x}{x+1}+\dfrac{x-1}{x}}{\dfrac{x}{x+1}-\dfrac{x-1}{x}}$

19. $\dfrac{\dfrac{m^2}{m^2-n^2}-1}{\dfrac{mn}{m-n}-n}$

20. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x}{x-y}-\dfrac{x}{x+y}}$

21. $\dfrac{\dfrac{x-m}{x-n}-\dfrac{x-n}{x-m}}{\dfrac{1}{x-m}-\dfrac{1}{x-n}}$

22. $\dfrac{\dfrac{1}{x-2}-\dfrac{4}{x^2-x-2}}{\dfrac{x}{x-5+\dfrac{6}{x}}-\dfrac{x}{x-2-\dfrac{3}{x}}}$

67. Equations Involving Fractions. In many cases, an equation containing fractions can be solved most easily by *clearing the equation of fractions* at the start. Since multiplying both members of an equation by the same quantity usually results in an equivalent equation, one needs only to multiply both sides of a fractional equation by the lowest common denominator of the fractions. This results in an equivalent equation *without fractions;* so the usual methods for solving equations can be applied.

Example 1. Solve: $\dfrac{x-3}{4}+\dfrac{2x}{3}=\dfrac{9}{10}$

Solution: The lowest common denominator of the fractions that appear in this equation is $2 \cdot 2 \cdot 3 \cdot 5 = 60$. Multiplying both sides of the equation by 60, one has

$$15x - 45 + 40x = 54$$

$$\text{T(45)}, \qquad 55x = 99$$

$$x = \tfrac{99}{55} = \tfrac{9}{5}$$

Example 2. $\dfrac{3}{x-1}+\dfrac{5}{x-2}-\dfrac{8}{x-3}=0$

Solution: The L.C.D. is $(x-1)(x-2)(x-3)$. Multiplication by the L.C.D. gives

$$3(x-2)(x-3)+5(x-1)(x-3)-8(x-1)(x-2)=0$$

Note that it is most convenient to keep the L.C.D. in factored form.

Then $\quad 3(x^2-5x+6)+5(x^2-4x+3)-8(x^2-3x+2)=0$

or $\quad 3x^2-15x+18+5x^2-20x+15-8x^2+24x-16=0$

Collecting, $\quad -11x+17=0 \quad$ or $\quad x=\frac{17}{11}$

Example 3. Solve for x and y:

$$\frac{1}{x-1} - \frac{1}{y-2} = 0 \tag{1}$$

$$\frac{2}{x-3} - \frac{1}{y-1} = 0 \tag{2}$$

Solution: Multiplying both sides of (1) by $(x - 1)(y - 2)$,

$$(y - 2) - (x - 1) = 0 \qquad \text{or} \qquad y - x - 1 = 0 \tag{3}$$

Multiplying both sides of (2) by $(x - 3)(y - 1)$,

$$2(y - 1) - (x - 3) = 0 \qquad \text{or} \qquad 2y - x + 1 = 0 \tag{4}$$

Subtracting (3) from (4),

$$y + 2 = 0 \qquad \text{or} \qquad y = -2$$

Substituting $y = -2$ in (3),

$$-2 - x - 1 = 0 \qquad \text{and} \qquad x = -3$$

Checking by substitution of $x = -3$, $y = -2$ in the original equations,

$$\frac{1}{-3-1} - \frac{1}{-2-2} = 0, \qquad -\tfrac{1}{4} + \tfrac{1}{4} \equiv 0, \qquad \text{check.}$$

$$\frac{2}{-3-2} - \frac{1}{-2-1} = 0, \qquad -\tfrac{1}{3} + \tfrac{1}{3} \equiv 0, \qquad \text{check.}$$

Sometimes it is best to eliminate one unknown before clearing of fractions. This is often the case when clearing of fractions would result in equations containing the product xy or other second-degree terms.

Example 4. Solve: $$\frac{1}{7x-3} - \frac{\frac{1}{3}}{y-1} = 0 \tag{1}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{10}{21} \tag{2}$$

Here (1) can be cleared of fractions without obtaining a second-degree equation; but (2) should not be cleared, since it would become

$$21y + 21x = 10xy,$$

a second-degree equation. Multiplying both members of (1) by the L.C.D.,

$$y - 1 - \tfrac{1}{3}(7x - 3) = 0$$

$$\text{M}(3), \qquad 3y - 3 - 7x + 3 = 0$$

$$3y = 7x \qquad \text{or} \qquad y = \tfrac{7}{3}x$$

Substituting in (2), $\frac{1}{x} + \frac{1}{\frac{7}{3}x} = \frac{10}{21}$

M$(21x)$, $21 + 9 = 10x$ or $x = 3$

Substituting in (2), $\frac{1}{3} + \frac{1}{y} = \frac{10}{21}$

M$(21y)$, $7y + 21 = 10y$

so that $3y = 21$ or $y = 7$

Checking in (1), $\frac{1}{21-3} - \frac{\frac{1}{3}}{7-1} = 0$ or $\frac{1}{18} - \frac{\frac{1}{3}}{6} \equiv 0$

Example 5. Solve:

$$\frac{2}{x} + \frac{3}{y} = 11 \qquad (1)$$

$$\frac{1}{x} - \frac{1}{y} = 1 \qquad (2)$$

Here neither equation can be cleared of fractions without obtaining second-degree terms. However, one can eliminate $\frac{1}{y}$ (and therefore y) by multiplying (2) by 3 and adding the result to (1), as follows:

$$\frac{3}{x} - \frac{3}{y} = 3 \qquad \text{(2), multiplied by 3}$$

$$\frac{2}{x} + \frac{3}{y} = 11 \qquad (1)$$

Adding, $\frac{5}{x} = 14$

M(x), $5 = 14x$

D(14), $x = \frac{5}{14}$

Substituting $x = \frac{5}{14}$ in (2), $\frac{1}{\frac{5}{14}} - \frac{1}{y} = 1$

or $\frac{14}{5} - \frac{1}{y} = 1$

M$(5y)$, $14y - 5 = 5y$

$y = \frac{5}{9}$

Checking, $\frac{2}{\frac{5}{14}} + \frac{3}{\frac{5}{9}} = 11$

M(5), $28 + 27 \equiv 55$, check

EXERCISES (67)

Solve the following:

1. $\frac{x+1}{4} + 3 = \frac{x+5}{2}$

2. $\frac{m+21}{9} = \frac{m-7}{2}$

3. $\frac{4x-1}{5} - \frac{2x+3}{4} = 2$

4. $\frac{3x-4}{7} - 3 = \frac{2x+5}{3}$

5. $\frac{c+4}{c} = \frac{c+9}{c+3}$

6. $\frac{2m}{m+2} = \frac{m}{m+1} + 1$

7. $\frac{1}{2} + \frac{4}{5x-5} = \frac{x}{2x-6}$

8. $\frac{y+5}{y-3} + \frac{4}{y-3} = 5$

9. $\frac{1}{x^2+9} = \frac{1}{x^2+7x+12}$

10. $\frac{2x+1}{x^2-4} = \frac{1}{x+2}$

11. $\frac{x}{3} - y = \frac{5}{3}$

$x - \frac{y}{3} = \frac{7}{3}$

12. $\frac{6x-1}{4} - y = 2$

$6x - 5y = 3$

13. $7x - 4y = 17$

$\frac{x+1}{2} + \frac{y-1}{3} = 15$

14. $\frac{x+y}{3} = \frac{3x-7}{2}$

$\frac{3x-2y}{5} = \frac{x-y}{2} + \frac{3}{10}$

15. $\frac{1}{x-1} + \frac{1}{y+1} = 0$

$\frac{1}{2x-3} + \frac{1}{y} = 0$

16. $\frac{x+1}{x} = \frac{y}{y+2}$

$\frac{x+5}{x+2} = \frac{y}{y-2}$

17. $\frac{x-4}{y+4} = \frac{x}{y}$

$\frac{x-3}{y-2} = \frac{x-3}{y}$

18. $\frac{1}{x} + \frac{1}{y} = 5$

$\frac{1}{x} - \frac{1}{y} = 1$

19. $\frac{1}{m} + \frac{1}{n} = 8$

$\frac{2}{m} + \frac{3}{n} = 25$

20. $\frac{2}{x} - \frac{1}{y} = 7$

$\frac{1}{x} + \frac{1}{y} = 5$

21. $\frac{5}{x} + \frac{2}{y} = 6$

$\frac{3}{x} + \frac{4}{y} = 12$

22. $\frac{3}{x} + \frac{2}{y} = 7$

$\frac{1}{x} - \frac{1}{y} = 11$

23. $\frac{2}{y} - \frac{1}{x} = 3$

$\frac{1}{y} + \frac{2}{x} = 11$

24. $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$

$\frac{3}{x} - \frac{8}{y} = 5$

25. $\frac{2}{y+2} - \frac{5}{3x+5} = 0$

$\frac{5}{x} + \frac{3}{y} = 9$

26. $\frac{1}{x} + \frac{1}{y} = 5$

$\frac{2}{x} + \frac{1}{z} = 10$

$\frac{1}{y} - \frac{1}{z} = -2$

68. Fractional Rates. If a pump requires 7 days to fill a 10,000-barrel tank, its average rate of pumping is $\frac{10{,}000 \text{ barrels}}{7 \text{ days}}$, or $1{,}428\frac{4}{7}$ barrels per day. If it fills a *certain tank* (size unspecified) in 7 days, its rate is $\frac{1}{7}$ *of that tank* per day. Thus $\frac{1}{7}$ (per day) is its *fractional rate.* If the pump fills the tank in x days, its fractional rate is $\frac{1}{x}$ (per day), or the *reciprocal of the time required.* (The reciprocal of a quantity is obtained by dividing it *into* the number 1. Thus $\frac{1}{x}$ is the reciprocal of x and $\frac{1}{x+2}$ the reciprocal of $x + 2$.)

A man who can complete a certain piece of work in 15 days will complete $\frac{2}{15}$ of it in 2 days, or $\frac{11}{15}$ of it in 11 days. This reasoning leads to the following rule:

> ***The fraction of the work accomplished equals the fractional rate multiplied by the time, if a constant rate of working is assumed.***

The maximum possible value of the fraction of work accomplished is unity, representing completion.

Example 1. One pump will fill a tank in 7 days, while another (alone) will fill it in 5 days. How many days will be required for the two pumps (together) to fill it?

Solution: The fractional rates for the two pumps are $\frac{1}{7}$ and $\frac{1}{5}$, respectively. If x days are required for them to fill the tank, they accomplish $\frac{x}{7}$ and $\frac{x}{5}$ of the task, respectively. Then

$$\frac{x}{7} + \frac{x}{5} = 1$$

$$\text{M}(35), \qquad 5x + 7x = 35$$

$$x = \frac{35}{12} = 2\frac{11}{12} \text{ (days)}$$

Example 2. A man can complete a certain manufacturing process in 20 days. On one occasion he works for 8 days, becomes ill, and is replaced by another man who finishes the job in 15 more days. How long would it take the second man to complete the entire process?

The fractional rate of the first man is $\frac{1}{20}$; hence he completes $\frac{8}{20}$ of the job in 8 days. If x is the number of days in which the second man

alone could complete the job, his fractional rate is $\frac{1}{x}$, and in 15 days he can do $\frac{15}{x}$ of the work. But the entire job is to be done; hence

$$\frac{8}{20} + \frac{15}{x} = 1$$

$$\text{M}(20x), \qquad 8x + 300 = 20x$$

$$12x = 300$$

$$x = 25 \text{ (days)}$$

Example 3. Two pumps (together) fill a certain reservoir in 25 hr. If (instead) one pump were operated alone for 15 hr. and then replaced by the other pump, the second would have to be operated 45 hr. in order to complete the job. How long would each pump require for the full job?

Solution: Let x and y be the respective amounts of time the individual pumps would require, so that $\frac{1}{x}$ and $\frac{1}{y}$ are their fractional rates.

Then $$\frac{25}{x} + \frac{25}{y} = 1 \qquad (1)$$

Also, $$\frac{15}{x} + \frac{45}{y} = 1 \qquad (2)$$

Multiplying (2) by 5, $$\frac{75}{x} + \frac{225}{y} = 5 \qquad (3)$$

Multiplying (1) by 3, $$\frac{75}{x} + \frac{75}{y} = 3 \qquad (4)$$

Subtracting, $$\frac{150}{y} = 2 \qquad \text{or} \qquad y = 75 \text{ (hours)}$$

Substituting in (1), $$\frac{25}{x} + \frac{25}{75} = 1$$

$$\text{M}(75x), \qquad (75)(25) + 25x = 75x$$

$$\text{D}(25), \qquad 75 + x = 3x$$

$$x = 37.5 \text{ (hours)}$$

Thus, the first pump would require 37.5 hr.; the second, 75 hr.

Example 4. A pump can fill a tank in 36 min., and a drain can empty it in 1 hr. If the drain is left open, how long will the pump require to fill the tank?

$$\frac{x}{36} - \frac{x}{60} = 1$$

$$\text{M}(180), \qquad 5x - 3x = 180$$

$$2x = 180$$

$$x = 90 \text{ min.}$$

PROBLEMS (68)

1. One man can mow a lawn in 2 hr. 20 min., while his son can do it in 3 hr. How long will it require the two, working together?

2. An airplane can fly from Boston to Chicago in 15 hr. Another can fly from Chicago to Boston in 10 hr. If they leave Boston and Chicago at the same time, how soon will they meet?

3. The rate of one pump is 50% more than that of another. Together they can fill a tank in 12 hr. How long would it require each separately?

4. Two pumps can empty a tank in 90 min. With the aid of a third pump, they can empty it in 40 min. How long would it require the third pump (alone)?

5. Two pumps (together) can fill a certain tank in 10 min. If one pump is reversed so that it tends to drain the tank, 1 hr. will be required to fill the tank. How long would be required by each pump (alone)?

6. Two men, working together, completed a job in 15 days. One of the men was working only half as fast as he could. Had he done his best, the job would have taken the two men only 12 days. How long would it take each man (alone) to complete the job, working at the same rate he maintained during the 15 days?

7. A tank can be emptied by one drain in 15 min., or by another in 10 min. If the tank is filled only two-thirds full, how long would it take the two drains to empty it?

8. A man works for 15 days on a job, then he is joined by another man, and they finish the work in 10 more days. The second man could have done the job alone in 60 days. How long would it take the first man to do the job?

9. A submarine is equipped with three diesel engines, one large engine and two small ones. With all engines going, the supply of fuel would last 72 cruising hours, but the same amount of fuel would run the large engine for 168 hr. How long would the fuel supply operate *one* of the small engines?

10. In Prob. 9, how long would the supply of fuel operate the large engine and one small one (simultaneously)?

11. A man can do a piece of work in 20 days, and his eldest son can do it in 25 days. They work together for 7 days; then they are joined by a younger son, with whose help they finish the work in 3 more days. How long would the entire job take the younger son?

12. If, on the average, a hen and a half lays an egg and a half in a day and a half, how long, on the average, would it require one hen to lay one egg?

13. Obtain a precise answer for Prob. 7, page 98. Observe that the graphical solution is necessary before an equation can be obtained.

REVIEW QUESTIONS

1. When is a fraction in its lowest terms?
2. How does one determine the L.C.D.?
3. When is it best to eliminate one of two unknowns *before* clearing the equations of fractions?
4. What is the reciprocal of a quantity?
5. State the fundamental principle involved in problems that have to do with fractional rates.

CHAPTER 9

SQUARE ROOTS AND RADICALS

Consider the equation $x^2 = 25$. Transposing 25,

$$x^2 - 25 = 0$$

Factoring, $$(x - 5)(x + 5) = 0$$

This shows that $x = 5$ and $x = -5$ are the *two* roots of the equation $x^2 = 25$.

Checking, $5^2 \equiv 25$ and $(-5)^2 \equiv 25$

69. Square Roots of Numbers. The two roots (5 and -5) of the equation $x^2 = 25$ are called the *square roots* of the *number* 25. Any positive number has two square roots, one positive and one negative, of the same size. The *radical** sign $\sqrt{\ }$ is used to indicate the positive or *principal* square root of a number. The quantity under the radical sign is called the *radicand.* For example, in the relation $\sqrt{25} = 5$, the radicand is 25.

Although the number 25 has *two* square roots 5 and -5, the quantity $\sqrt{25}$ (*radical* 25) equals *only* 5. Strictly, then, the radical sign should not be referred to as a "square-root" sign. The negative square root of 25 can be indicated by $-\sqrt{25}$ (minus radical 25), which equals -5, since $\sqrt{25} = 5$. Thus the (indicated) square roots of 25 are $\pm\sqrt{25}$ (read, "plus *and* minus radical 25"), which are equivalent to ± 5. Whenever one refers to *the* square root of a number, he means only the principal (or positive) square root, as indicated by the radical sign.

The square roots of numbers have many practical applications, particularly in connection with quadratic equations. When a quadratic equation cannot be solved by factoring (and most of them cannot), it can be solved most conveniently by a method described in the next chapter, a method that involves the use of the indicated *square roots* of numbers. Thus, it is important to understand the methods of multiplying, dividing, adding, and subtracting radicals, as well as the methods of determining their numerical values.

* From the Latin word *radix*, which means "root."

70. Multiplication and Division of Radicals. In multiplying and dividing radicals, write the product or quotient of the radicands under a single radical sign.

Example 1. $\sqrt{6} \cdot \sqrt{4} = \sqrt{24}$

Example 2. $\sqrt{12} \cdot \sqrt{3} = \sqrt{36} = 6$

Example 3. $\sqrt{5a} \cdot \sqrt{ab} = \sqrt{5a^2b}$

Example 4. $\dfrac{\sqrt{48}}{\sqrt{12}} = \sqrt{\dfrac{48}{12}} = \sqrt{4} = 2$

Example 5. $\dfrac{\sqrt{6ab^3}}{\sqrt{2a^2b}} = \sqrt{\dfrac{6ab^3}{2a^2b}} = \sqrt{\dfrac{3b^2}{a}}$

In order to reduce a radical expression to its simplest form, it is necessary

1. To remove from the radicands all factors that are perfect squares.
2. To remove all fractions from the radicands.
3. To remove all radical signs from the denominators of fractions.

Example 6. $\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$, in simplest form.

The square root of an expression that is raised to a power is obtained by dividing the exponent by 2. Fractional exponents can be avoided by factoring out and removing only even powers.

Example 7. $\sqrt{16a^2b^4c^6} = \sqrt{16} \cdot \sqrt{a^2} \cdot \sqrt{b^4} \cdot \sqrt{c^6} = 4ab^2c^3$

Example 8. $\sqrt{12x^4y^5} = \sqrt{4} \cdot \sqrt{3} \cdot \sqrt{x^4} \cdot \sqrt{y^4} \cdot \sqrt{y} = 2x^2y^2\sqrt{3y}$

One simplifies a fractional radical by multiplying both numerator and denominator by a *rationalizing factor*, a factor that makes the denominator a perfect square.

Example 9. $\sqrt{\dfrac{2}{3}} = \sqrt{\dfrac{2 \times 3}{3 \times 3}} = \dfrac{\sqrt{6}}{\sqrt{9}} = \dfrac{1}{3}\sqrt{6}$

Example 10. $\sqrt{\dfrac{7}{8}} = \sqrt{\dfrac{14}{16}} = \dfrac{\sqrt{14}}{\sqrt{16}} = \dfrac{1}{4}\sqrt{14}$

Example 11. $\dfrac{6x^2y}{\sqrt{18x^3y}} = \dfrac{6x^2y\sqrt{2xy}}{\sqrt{36x^4y^2}} = \dfrac{6x^2y\sqrt{2xy}}{6x^2y} = \sqrt{2xy}$

EXERCISES (70)

Reduce to simplest form:

1. $\sqrt{32}$
2. $\sqrt{12}$
3. $\sqrt{75}$
4. $\sqrt{900}$
5. $\sqrt{48}$
6. $\sqrt{8a^5b^6}$
7. $\sqrt{3} \cdot \sqrt{27}$
8. $\sqrt{2} \cdot \sqrt{18}$
9. $\sqrt{3} \cdot \sqrt{13}$
10. $\sqrt{9x} \cdot \sqrt{x}$
11. $\sqrt{32x^3} \cdot \sqrt{\dfrac{x}{2}}$
12. $\dfrac{\sqrt{50}}{\sqrt{2}}$
13. $\dfrac{\sqrt{180}}{\sqrt{5}}$
14. $\sqrt{\frac{2}{5}}$
15. $\sqrt{\frac{5}{6}}$
16. $\sqrt{\frac{3}{20}}$
17. $\sqrt{\frac{5}{21}}$
18. $\sqrt{5\frac{3}{7}}$
19. $\sqrt{2\frac{1}{6}}$
20. $\dfrac{\sqrt{5}}{\sqrt{14}}$
21. $\dfrac{\sqrt{45}}{\sqrt{8}}$
22. $\dfrac{5}{\sqrt{12}}$
23. $\dfrac{5\sqrt{2}}{3\sqrt{75}}$
24. $\dfrac{2\sqrt{3}}{5\sqrt{20}}$
25. $\sqrt{\dfrac{8m^5n^2}{75a^3m}}$
26. $\sqrt{\dfrac{18x^3y^2}{12z^3}}$

71. Addition and Subtraction of Radicals. Quadratic radicals that have the same radicand are called *similar* (or *like*) radicals. For example, $2\sqrt{3}$, $\frac{2}{7}\sqrt{3}$, and $3a^2\sqrt{3}$ are similar radicals, while $2\sqrt{3}$, $3\sqrt{2}$, and $5\sqrt{7}$ are *unlike* radicals. Similar radicals may be combined according to the rules for addition and subtraction of similar terms. Unlike radicals cannot be combined by addition and subtraction unless they can be reduced to similar radicals.

Example 1. $\sqrt{27} + \sqrt{75} - \sqrt{12} = 3\sqrt{3} + 5\sqrt{3} - 2\sqrt{3} = 6\sqrt{3}$

Example 2.

$$\begin{aligned}
\sqrt{18} + \sqrt{\tfrac{27}{6}} - 2\sqrt{\tfrac{1}{3}} + \sqrt{4\tfrac{1}{2}} &= \sqrt{9} \cdot \sqrt{2} + \sqrt{\tfrac{9}{2}} - 2\sqrt{\tfrac{3}{9}} + \sqrt{\tfrac{9}{2}} \\
&= 3\sqrt{2} + \sqrt{\frac{9 \times 2}{4}} - \frac{2}{3}\sqrt{3} + \sqrt{\frac{9 \times 2}{4}} \\
&= 3\sqrt{2} + \tfrac{3}{2}\sqrt{2} - \tfrac{2}{3}\sqrt{3} + \tfrac{3}{2}\sqrt{2} \\
&= 6\sqrt{2} - \tfrac{2}{3}\sqrt{3}
\end{aligned}$$

EXERCISES (71)

Combine:

1. $\sqrt{50} - \sqrt{18}$
2. $\sqrt{75} + \sqrt{27}$
3. $\sqrt{20} + \sqrt{45}$
4. $\sqrt{108} - \sqrt{12}$
5. $\sqrt{125} + \sqrt{45}$
6. $5\sqrt{18} - 3\sqrt{8}$
7. $5\sqrt{63} + 3\sqrt{28}$
8. $\sqrt{675} + \sqrt{432}$
9. $\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{2}}$

10. $\sqrt{\frac{9}{2}} - \sqrt{\frac{2}{9}} - \sqrt{\frac{1}{18}}$

11. $2\sqrt{5\frac{1}{3}} - 3\sqrt{2\frac{2}{3}}$

12. $27\sqrt{\frac{1}{18}} - \dfrac{5}{\sqrt{50}} + 15\sqrt{\frac{2}{5}}$

13. $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} - \sqrt{\dfrac{1}{xy}}$

14. $2\sqrt{\frac{14}{3}} - 9\sqrt{\frac{6}{7}} + 5\sqrt{\frac{2}{21}}$

15. $m\sqrt{na^2} + a\sqrt{m^2n} + n\sqrt{n}$

72. Multiplication of Multinomials That Involve Radicals. In multiplying multinomials that contain radicals, first simplify the radicals, then proceed according to the rules observed in the multiplication of ordinary multinomials.

Example 1. Multiply $2\sqrt{3} + 3\sqrt{6} - \sqrt{24}$ by $5\sqrt{3}$.

Simplifying, $\quad 2\sqrt{3} + 3\sqrt{6} - 2\sqrt{6} = 2\sqrt{3} + \sqrt{6}$

Then $\quad 5\sqrt{3}\,(2\sqrt{3} + \sqrt{6}) = 10 \times 3 + 5\sqrt{18} = 30 + 15\sqrt{2}$

Example 2. Multiply $\sqrt{50} - \sqrt{20}$ by $\sqrt{45} + \sqrt{32}$.
Simplifying,

$$\begin{aligned}(\sqrt{50} - \sqrt{20})(\sqrt{45} + \sqrt{32}) &= (5\sqrt{2} - 2\sqrt{5})(3\sqrt{5} + 4\sqrt{2}) \\ &= 15\sqrt{10} - 30 + 40 - 8\sqrt{10} \\ &= 10 + 7\sqrt{10}\end{aligned}$$

Example 3. Is $x = 2 + \sqrt{2}$ a root of the equation $x^2 - 5x + 7 = 0$?

Substituting, $\quad (2 + \sqrt{2})^2 - 5(2 + \sqrt{2}) + 7 = 0$

Now, $\quad (2 + \sqrt{2})^2 = 4 + 4\sqrt{2} + 2 = 6 + 4\sqrt{2}$

so that the equation becomes

$$6 + 4\sqrt{2} - 10 - 5\sqrt{2} + 7 = 0 \qquad \text{or} \qquad 3 - \sqrt{2} = 0$$

This does not check, hence $x = 2 + \sqrt{2}$ is not a root of $x^2 - 5x + 7 = 0$.

EXERCISES (72)

Express in simplest form:

1. $\sqrt{2}\,(\sqrt{6} + \sqrt{18})$

2. $\sqrt{2}\,(\sqrt{6} + 2)$

3. $\sqrt{12}\,(\sqrt{3} - \sqrt{6} + \sqrt{2})$

4. $\sqrt{15}\,(\sqrt{125} - \sqrt{10} - \sqrt{21})$

5. $\sqrt{10}\,(5\sqrt{2} - \sqrt{10} - 2\sqrt{5})$

6. $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$

7. $2\sqrt{6}\,(9\sqrt{3} + 2\sqrt{6} + 5\sqrt{2})$

8. $(\sqrt{2} + \sqrt{3})^2$

9. $(\sqrt{5} + 3\sqrt{3})(5\sqrt{5} - 2\sqrt{3})$

10. $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$

11. $(2\sqrt{3} + 3\sqrt{2})^2$
12. $(\sqrt{18} + 2\sqrt{27})(\sqrt{75} - \sqrt{8})$
13. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$
14. $(\sqrt{a} + \sqrt{b})^2$
15. $(\sqrt{x} + \sqrt{y})^2\sqrt{xy}$
16. $(\sqrt{72} + 2\sqrt{20})(3\sqrt{45} - 5\sqrt{18})$

Determine the value of

17. $x^2 + 3x + 1$, when $x = 1 - \sqrt{2}$
18. $x^2 - 3x + 5$, when $x = 1 + \sqrt{3}$
19. $(x - 2)(x - 3)$, when $x = 1 - \sqrt{3}$
20. $x^2 - 4x + 1$, when $x = \sqrt{3} + 2$
21. $x^2 + 2xy + y^2$, when $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$
22. $x^2 - 2xy + y^2$, when $x = 1 - \sqrt{2}$ and $y = 2 - \sqrt{2}$
23. $x^2 + 3xy + y^2$, when $x = \sqrt{3} + \sqrt{2}$ and $y = \sqrt{3} - \sqrt{2}$
24. $x^3 - 3x^2 + 2x + 5$, when $x = 2 - \sqrt{3}$

73. Division of Multinomials That Involve Radicals. In general, it is best to multiply both dividend and divisor by a radical that will eliminate the radical in the divisor. Although this process (called rationalizing the divisor) is not necessary when the divisor is a monomial, it tends to simplify the solution.

Example 1. $\dfrac{9\sqrt{2} - 6\sqrt{6} + 18\sqrt{3}}{3\sqrt{6}}$.

Multiplying both numerator and denominator by $\sqrt{6}$,

$$\frac{18\sqrt{3} - 36 + 54\sqrt{2}}{18} = \sqrt{3} - 2 + 3\sqrt{2} = \sqrt{3} - 2 + 3\sqrt{2}$$

When the divisor is a binomial, it can be rationalized by multiplying both dividend and divisor by the divisor with the sign of either of its terms changed.

Example 2. $$\frac{\sqrt{10}}{3\sqrt{2} - \sqrt{5}} = \frac{\sqrt{10}(3\sqrt{2} + \sqrt{5})}{(3\sqrt{2} - \sqrt{5})(3\sqrt{2} + \sqrt{5})}$$
$$= \frac{\sqrt{10}(3\sqrt{2} + \sqrt{5})}{18 - 5} = \frac{6\sqrt{5} + 5\sqrt{2}}{13}$$

Note that both numerator and denominator were multiplied by $3\sqrt{2} + \sqrt{5}$ and that, in the denominator,

$$(3\sqrt{2} - \sqrt{5})(3\sqrt{2} + \sqrt{5}) = 13$$

In general, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.

EXERCISES (73)

Express in simplest form:

1. $\dfrac{18\sqrt{14} - 4\sqrt{10} + 2\sqrt{6}}{2\sqrt{2}}$

2. $\dfrac{10\sqrt{6} - 25\sqrt{33} - 15\sqrt{15}}{5\sqrt{3}}$

3. $\dfrac{9\sqrt{3} + 3\sqrt{5} - 6\sqrt{2}}{\sqrt{3}}$

4. $\dfrac{18\sqrt{5} + 36\sqrt{11} + 12\sqrt{7}}{\sqrt{6}}$

5. $\dfrac{5\sqrt{6} - 15\sqrt{26} - 20\sqrt{14}}{\sqrt{10}}$

6. $\dfrac{15\sqrt{15} - 2\sqrt{2} - 6\sqrt{6}}{\sqrt{30}}$

7. $\dfrac{5\sqrt{3} + 3\sqrt{7} - 3}{4\sqrt{3}}$

8. $\dfrac{9\sqrt{13} - 2\sqrt{19} + 2\sqrt{5}}{3\sqrt{6}}$

9. $\dfrac{2}{\sqrt{2} + \sqrt{3}}$

10. $\dfrac{5}{1 - \sqrt{3}}$

11. $\dfrac{1 - \sqrt{6}}{\sqrt{2} + \sqrt{3}}$

12. $\dfrac{18 + 2\sqrt{7}}{1 - 3\sqrt{7}}$

13. $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{3} - \sqrt{2}}$

14. $\dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$

15. $\dfrac{5\sqrt{2}}{2\sqrt{5} - 5\sqrt{2}}$

16. $\dfrac{83 - \sqrt{35}}{3\sqrt{5} + \sqrt{7}}$

17. $\dfrac{1}{\sqrt{m} - \sqrt{n}}$

18. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$

19. $\dfrac{\sqrt{15}}{2\sqrt{3} + 9\sqrt{2}} \cdot \dfrac{\sqrt{6}}{2\sqrt{3} - 9\sqrt{2}}$

20. $\dfrac{5 + 9\sqrt{3}}{2 + \sqrt{3}} \cdot \dfrac{\sqrt{3}}{3 + \sqrt{3}}$

21. $\dfrac{2\sqrt{3}}{1 - 3\sqrt{3}} \cdot \dfrac{\sqrt{3}}{1 - \sqrt{3}}$

22. $\dfrac{5\sqrt{3} - 3\sqrt{2}}{\sqrt{2} + \sqrt{3}} \cdot \dfrac{2\sqrt{3} + 3\sqrt{2}}{\sqrt{2} - \sqrt{3}}$

23. $\dfrac{9 - \sqrt{6}}{\sqrt{6} + 1} \cdot \dfrac{1}{\sqrt{2} - \sqrt{3}}$

24. $\dfrac{5 + 9\sqrt{3}}{2 + \sqrt{3}} \div \dfrac{3 + \sqrt{3}}{\sqrt{3}}$

74. Evaluation of Radicals. The values of $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, etc., can be determined at sight, since these square roots are small integers. The value of $\sqrt{667,489} = 817$, however, cannot easily be determined by inspection. Likewise, radicals whose values are irrational,* such as $\sqrt{5}$, $\sqrt{7}$, cannot be evaluated by inspection. In fact, their precise values cannot be expressed in decimal form; though any desired degree of accuracy can be obtained by expressing their approximate values to more decimal

* A *rational* number is one that can be expressed as the ratio of two integers, an *irrational* number is one that cannot.

places. For example, $\sqrt{5} \doteq 2.2$, approximately. More accurately, $\sqrt{5} \doteq 2.236$; or, still more accurately, $\sqrt{5} \doteq 2.23607$. That this last relation is only approximate can be verified by multiplication:

$$2.23607 \times 2.23607 = 5.0000090449, \text{ exactly.}$$

Thus we see that the principal square root of 5, represented by $\sqrt{5}$, is slightly *smaller* than 2.23607. Now,

$$2.23606 \times 2.23606 = 4.9999643236$$

showing that $\sqrt{5}$ is larger than 2.23606, or that it is *between* 2.23606 and 2.23607. By going another decimal place, we can learn that $\sqrt{5}$ is between 2.236067 and 2.236068, etc. There is no place to stop in this process—at some point we must be satisfied with an *approximate* value. *In general, radicals cannot be expressed exactly in decimal form.*

The arithmetic method of extracting the positive square root of a number should be familiar to the student from earlier courses in mathematics. Two examples will be given, in order to assist him in recalling the process to mind.

Example 1. Evaluate $\sqrt{667.489}$.

Solution: First, divide the radicand into groups of two digits each, proceeding in each direction from the decimal point. Next, determine the largest perfect square number which does not exceed the first group of two digits. In this example the largest square in 66 is $64 = 8^2$; so write the square root 8 above the first group, as shown, and write 64 below the first group, 66. Then subtract, obtaining 2. Now, bring down the next group (74), obtaining 274. Obtain a trial divisor by multiplying the square root thus far obtained (8) by 2, and write it at the left of 274. Divide this trial divisor (16) into 27, that is, into 274 with the last digit (4) removed. Write the result (1) as the last digit of the divisor (and also as the second digit of the square root), then multiply the divisor by it (1) and subtract the result (161) from 274, obtaining 113.

$$\begin{array}{rr}
 & 8\ \ 1\ \ 7. \\
 & \sqrt{66\ 74\ 89.} \\
 & 64 \\
\mathbf{161} & 2\ 74 \\
 & 1\ 61 \\
\mathbf{1627} & 1\ 13\ 89 \\
 & 1\ 13\ 89
\end{array}$$

Bring down the next group of two digits, multiplying the square root thus far obtained (81) by 2, to find a new trial divisor (162), which is written in the place shown. Divide the trial divisor into 1,138 (*i.e.*, 11,389 with the last digit removed), obtaining 7, or approximate the

result by dividing 16 into 113. Write the result (7) both as the third digit of the square root and as the last digit of the divisor; then multiply the latter by it and subtract as before. In general, this process is continued until there is no remainder, or until a sufficient number of decimal places have been obtained. The latter is the case when the radicand is not a perfect square.

Example 2. Evaluate $\sqrt{5228.55}$, accurate to hundredths.

Solution: Proceed as in Example 1, continuing the process until the square root has been carried to three decimal places, then "round off" to hundredths. The square root will contain one digit for each group in the radicand.

	$7\quad 2\,.\,3\quad 0\quad \cancel{9}^{8}$
	$\sqrt{52\ 28.55\ 00\ 00}$
	49
142	3 28
	2 84
1443	44 55
	43 29
14460	1 26 00
	0 00 00
$1446\cancel{9}^{8}$	1 26 00 00
	1 15 68 64
	10 31 36

Four things should be observed in connection with this example. (1) There is a remainder, and one could keep on computing the result to more decimal places indefinitely. Thus the result cannot be expressed exactly by any number of decimal places; it must be "rounded off" somewhere. (2) At one stage the trial divisor could not be divided into the remainder; so the digit 0 was recorded, and another group was brought down. (3) In the last step, the digit obtained by means of the trial divisor was too large, producing a negative remainder; hence the next smaller digit (8) was used. (4) The result was carried to five figures (not decimal places) in order to be sure of accuracy to four figures. The result is $\sqrt{5228.55} \doteq 72.31$.

EXERCISES (74)

Evaluate the following, to four figures (not four decimal places):

1. $\sqrt{5}$	2. $\sqrt{27}$	3. $\sqrt{62.184}$	4. $\sqrt{.0274}$
5. $\sqrt{72{,}859}$	6. $\sqrt{14}$	7. $\sqrt{334.621}$	8. $\sqrt{.382}$
9. $\sqrt{.627843}$	10. $\sqrt{.006}$	11. $\sqrt{12345678}$	12. $\sqrt{284.75}$
13. $\sqrt{.01762}$	14. $\sqrt{627}$	15. $\sqrt{12.74}$	16. $\sqrt{224.7}$
17. $\sqrt{7.426}$	18. $\sqrt{41.72}$	19. $\sqrt{2}$	20. $\sqrt{.7}$

75. Tables of Square Roots. A considerable amount of labor is involved in the calculation of square roots by the arithmetic method. For this reason, the arithmetic method should be used

only when more convenient methods will not produce the desired accuracy.

The principal square roots of the integers from 1 to 100 are given in Table 1, page 287, in which the values listed are accurate to four figures. By referring to the table, verify the following:

$$\sqrt{24} \doteq 4.899, \qquad \sqrt{96} \doteq 9.798, \qquad \sqrt{11} \doteq 3.317$$

In order to evaluate the square root of a number outside the range 1 to 100, it is necessary to express its value by means of a radicand within the desired range.

Example 1. Evaluate $\sqrt{270}$, using the table of square roots.

Solution: The radicand 270 is outside the range of values listed in the table. But

$$\sqrt{270} = \sqrt{9 \times 30} = 3\sqrt{30}.$$

From the table,

$$\sqrt{30} \doteq 5.477;$$

hence

$$\sqrt{270} \doteq 3 \times 5.477 = 16.431 \doteq 16.43$$

Three things should be noticed in connection with this example:

1. The radicand was factored, so that the perfect square 9 could be replaced by its square root, *outside* the radical sign, leaving a radicand between 0 and 100.

2. A dot was written over the equal sign in the equation, $\sqrt{30} \doteq 5.477$. This dot indicates that the relation indicated is an approximate one, since 5.477 has been rounded off at four figures. The dot will be used in all indicated equalities that are not exact.

3. The result 16.431 was rounded off to four figures, 16.43. Since the value taken from the table, 5.477, is rounded off at four figures, the fifth significant figure in any value computed by means of it will be unreliable. Never carry more significant figures than are given in the table.

In rounding off numbers to four figures, always select the *nearest* number having four figures. For example, $2.46736 \doteq 2.467$ (to four figures), and $2.4676 \doteq 2.468$. Also $2.46751 \doteq 2.468$, since it is slightly nearer 2.468 than 2.467. When a number is *exactly* midway between the two nearest values, as when 5 is the (only) digit to be dropped, the nearest *even* (not odd) terminating digit should be selected. Thus, $2.4675 \doteq 2.468$, $2.4685 \doteq 2.468$, $2.4695 \doteq 2.470$, and $2.4605 \doteq 2.460$.

When it is desired to increase the radicand, any factor may be introduced into the radicand if, at the same time, its square root is written as a divisor outside the radical.

Example 2. Evaluate $\sqrt{.29}$.

Solution: $\sqrt{.29} = \frac{1}{10}\sqrt{29} \doteq .1 \times 5.385 = .5385$

Example 3. $\sqrt{1.64} = .1\sqrt{164} = .1\sqrt{4 \times 41} = .2\sqrt{41}$

From the table, $\sqrt{41} \doteq 6.403$; hence $\sqrt{1.64} = .2 \times 6.403 \doteq 12.81$.

Example 4. Evaluate $\sqrt{3\frac{1}{2}}$.

Solution: $\sqrt{3\frac{1}{2}} = \sqrt{\frac{7}{2}}$. Although $\sqrt{7}$ and $\sqrt{2}$ can be found from the tables, so that $\frac{\sqrt{7}}{\sqrt{2}} \doteq \frac{2.646}{1.414}$, the radical should be simplified before any square roots are obtained from the table. Thus

$$\sqrt{3\tfrac{1}{2}} = \sqrt{\tfrac{7}{2}} = \tfrac{1}{2}\sqrt{14} \doteq \tfrac{1}{2}(3.742) = 1.871$$

Note that without this simplification a division of 2.646 by 1.414 would have been necessary.

Example 5. Evaluate $\frac{2}{\sqrt{3} - \sqrt{2}}$.

Solution: As in the preceding example, it is best to simplify the expression before evaluating any square roots. Thus

$$\frac{2}{\sqrt{3} - \sqrt{2}} = \frac{2(\sqrt{3} + \sqrt{2})}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})} \doteq \frac{2(1.732 + 1.414)}{1}$$
$$= 2(3.146) \doteq 6.292$$

It is to be assumed, for the present, that the table cannot be used to determine the square root of a number *between* two integers listed in the table, such as 24.3. The method of obtaining the square roots of such numbers (called *interpolation*, or "placing between") will be explained in the next section.

EXERCISES (75)

Evaluate the following, using the table of square roots:

1. $\sqrt{77}$	2. $\sqrt{56}$	3. $\sqrt{120}$	4. $\sqrt{180}$	5. $\sqrt{148}$
6. $\sqrt{168}$	7. $\sqrt{117}$	8. $\sqrt{156}$	9. $\sqrt{297}$	10. $\sqrt{3,900}$
11. $\sqrt{6,600}$	12. $\sqrt{.67}$	13. $\sqrt{.92}$	14. $\sqrt{1.2}$	15. $\sqrt{1,120,000}$
16. $\sqrt{.240}$	17. $\sqrt{2\frac{3}{4}}$	18. $\sqrt{3\frac{3}{8}}$	19. $\sqrt{12\frac{3}{8}}$	20. $\sqrt{14\frac{2}{3}}$

21. $\dfrac{7}{\sqrt{5}+\sqrt{2}}$ 22. $\dfrac{5}{\sqrt{7}-\sqrt{3}}$ 23. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

24. $\dfrac{5\sqrt{2}}{2\sqrt{5}-5\sqrt{2}}$ 25. $\dfrac{1+\sqrt{\frac{1}{2}}}{1-\sqrt{\frac{1}{3}}}$ 26. $\dfrac{5-\sqrt{3}}{5+\sqrt{3}}$

76. Interpolation. The method of obtaining the square roots of numbers between those listed in the table will be illustrated by examples.

Example 1. Evaluate $\sqrt{85.5}$.

Solution: Though $\sqrt{85.5}$ does not appear in the table, its value can be determined with reasonable accuracy by assuming that, since 85.5 is midway between 85 and 86, $\sqrt{85.5}$ is midway between $\sqrt{85}$ and $\sqrt{86}$.

Now $\qquad \sqrt{86} \doteq 9.274$

and $\qquad \sqrt{85} \doteq 9.220$

$\qquad$ Difference $= \ .054$

This difference is called the *tabular difference.* Then

$$\sqrt{85.5} \doteq \sqrt{85} + .5(.054) \doteq 9.220 + .027 = 9.247$$

During interpolation, the values taken from the table are usually written without decimal points; *viz.*,

$$9274 - 9220 = 54 \qquad \text{and} \qquad .5 \times 54 = 27$$

Then $\qquad 9220 + 27 = 9247$

hence $\qquad \sqrt{85.5} \doteq 9.247$

Example 2. Evaluate $\sqrt{26.7}$.

Solution: $\sqrt{26.7}$ is between $\sqrt{26}$ and $\sqrt{27}$. From the table,

$$\sqrt{27} \doteq 5.196 \qquad \text{and} \qquad \sqrt{26} \doteq 5.099.$$

Then, dropping the decimal points during interpolation,

$$5196 - 5099 = 97$$

the tabular difference. Assuming that $\sqrt{26.7}$ is .7 of the way between $\sqrt{26}$ and $\sqrt{27}$, we add .7 of the tabular difference to the smaller of the two nearest values listed in the table. Thus

$$5099 + .7(97) = 5099 + 67.9 \doteq 5099 + 68 = 5167$$

Then $\qquad \sqrt{26.7} \doteq 5.167$

Remember that the values $\sqrt{27} \doteq 5.196$ and $\sqrt{26} \doteq 5.099$ were themselves rounded off to four figures in making up the table; hence no

value calculated from them will be reliable beyond four figures. In fact, the result obtained by interpolation will frequently be in error by one unit in the fourth figure. For this reason, round off all numbers at four significant figures, both in intermediate steps and in the result.

EXERCISES (76)

Evaluate by means of the square root table:

1. $\sqrt{28.5}$	2. $\sqrt{43.5}$	3. $\sqrt{32.7}$	4. $\sqrt{56.3}$
5. $\sqrt{86.2}$	6. $\sqrt{92.7}$	7. $\sqrt{62.4}$	8. $\sqrt{17.2}$
9. $\sqrt{11.25}$	10. $\sqrt{32.75}$	11. $\sqrt{41.23}$	12. $\sqrt{22.2}$
13. $\sqrt{10.7}$	14. $\sqrt{72.62}$	15. $\sqrt{31.4}$	16. $\sqrt{12.21}$

77. Modifying the Radicand. As was shown in Sec. 75, when the square root of a number is outside the range 0 to 100, it is necessary to express its value by means of a radicand within that range. Moreover, when interpolation is involved, the radicand must be greater than 10; else the results obtained by interpolation will be unreliable.

> ***The results of interpolation in the four-place table of square roots will be accurate to within one unit in the fourth figure, if the radicand is greater than 10.***

In order to see why interpolation sometimes yields inaccurate results, refer to Fig. 24, in which the value of $\sqrt{x}$ is plotted as a function of x for the range $x = 0$ to $x = 3$. Check the values of $\sqrt{1}$, $\sqrt{2}$, and $\sqrt{3}$ on this graph to familiarize yourself with it; then consider the following:

When one interpolates, it is assumed that a small portion of the graph of $\sqrt{x}$ is for practical purposes a straight line. Refer to the graph, and notice the dotted lines between the points for $x = 0$ and $x = 1$, $x = 1$ and $x = 2$, etc. The values obtained by interpolation lie on these lines and are thus in error by the amounts indicated by the deviation of the lines from the graph. Observe that this deviation decreases rapidly as x becomes larger. It becomes negligible, for practical purposes, at about $x = 10$.

Example 1. Evaluate $\sqrt{1.67}$.

Solution: Direct interpolation will probably lead to an inaccurate result in this case, since the radicand is smaller than 10.

From the table, $\sqrt{2} \doteq 1.414$

and $\sqrt{1} = 1.000$

The tabular difference is 414, and

$$.67 \times 414 \doteq 277$$

Then $1000 + 277 = 1277$, so that $\sqrt{1.67} \doteq 1.277$ is the (inaccurate) result of direct interpolation. Actually, $\sqrt{1.67} \doteq 1.292$. Refer to Fig. 24, observing that at $x = 1.67$ the separation between the straight

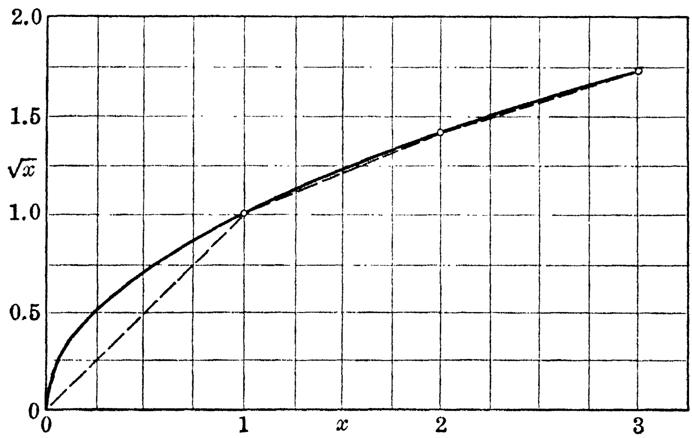


FIG. 24.—Graph of the function $\sqrt{x}$.

line and the graph of $\sqrt{x}$ is perceptible. This separation corresponds to the difference between 1.277 and 1.292.

In order to obtain a correct result by interpolation in this case, it is necessary to modify the radical so that the new radicand is between 10 and 100; *viz.*,

$$\sqrt{1.67} = \tfrac{1}{10}\sqrt{167} = \tfrac{2}{10}\sqrt{\tfrac{167}{4}} = \tfrac{1}{5}\sqrt{41.75}$$

Observe that this modification was performed in two steps, first multiplying the radicand by 100, then dividing it by 4.

Now, $\sqrt{42} \doteq 6.481$

and $\sqrt{41} \doteq 6.403$

The tabular difference is 78, and $.75 \times 78 \doteq 58$ (to the nearest integer) Then $6403 + 58 = 6461$, so that $\sqrt{41.75} \doteq 6.461$. Thus

$$\sqrt{1.67} = \frac{1}{5}\sqrt{41.75} \doteq \frac{6.461}{5} \doteq 1.292$$

correct to four figures.

In modifying a radical to produce a radicand between 10 and 100, we are not restricted to division of the radicand by exact factors, since we are prepared to interpolate. It is preferable to multiply or divide the radicand by only those perfect squares that can be handled *mentally*, without difficulty. These numbers include only 4, 9, and 100.

Check the following examples:**

$$\sqrt{6.7} = \frac{1}{3}\sqrt{9 \times 6.7} = \frac{1}{3}\sqrt{60.3} \doteq \frac{7.765}{3} \doteq 2.588$$

$$\sqrt{544} = 3\sqrt{\tfrac{544}{9}} \doteq 3\sqrt{60.44} \doteq 3 \times 7.774 \doteq 24.32$$

$$\sqrt{12{,}300} = 10\sqrt{123} = 20\sqrt{\tfrac{123}{4}} = 20\sqrt{30.75} \doteq 20 \times 5.545 \doteq 110.9$$

$$\sqrt{.0273} = \frac{1}{100}\sqrt{273} = \frac{2}{100}\sqrt{68.25} \doteq \frac{8.261}{50} \doteq .1652$$

Only the numbers 4, 9, 100 should be used, unless an exact factor (a perfect square) can be observed, in which case interpolation can be avoided. In most cases, exact perfect-square factors will not be found.

Examine the table of square roots, observing that the tabular differences become smaller as the numbers (and their square roots) increase. This means that interpolation will be less difficult and more accurate when the radicand is large. In choosing one of the factors 4, 9, or 100, make the choice for which the radicand (within the range 10 to 100, of course) is largest.

By means of the column headed $\sqrt{10N}$ in Table 1, the radicand can be modified properly by using only the factor 100. This column extends the range of the table to include radicands in the range 100 to 1,000. It is thus necessary only to shift the decimal point an even number of places to bring the radicand within the range 10 to 1,000 which is covered by the two columns. Because a table of this type is not always available, it is suggested that the instructor assign at least a few exercises to be done without the use of the $\sqrt{10N}$ column.

EXERCISES AND PROBLEMS (77)

Evaluate:

1. $\sqrt{4.2}$	2. $\sqrt{7.3}$	3. $\sqrt{170}$	4. $\sqrt{226}$
5. $\sqrt{241}$	6. $\sqrt{685}$	7. $\sqrt{2100}$	8. $\sqrt{847}$
9. $\sqrt{.037}$	10. $\sqrt{.694}$	11. $\sqrt{1.2312}$	12. $\sqrt{.000146}$
13. $\sqrt{7.15}$	14. $\sqrt{.0263}$	15. $\sqrt{44321}$	16. $\sqrt{612.9}$

17. Find the hypotenuse* of a right triangle whose other two sides are 31 and 47 ft., respectively, in length.

18. Find the third side of a right triangle whose hypotenuse is 22.3 ft. long, if the other side is 4.7 ft. long.

19. A guy wire 88 ft. long reaches from the top of a radio-antenna tower to a point on the ground 50.3 ft. from the base of the tower. If the wire is practically straight, how high is the tower?

20. The area of a square is 4,564 sq. in. How long is its diagonal?

21. Find the diagonal of a room whose dimensions in feet are 19.3, 17.2, and 9.6.

22. A flagpole 122 ft. high breaks at the base and falls directly toward a building 74 ft. away. How high on the side of the building will the top of the pole strike?

23. A picture is 16 by 20 in. in size, and its frame is 3 in. wide, on all sides. Find the distances from one corner of the picture to each of the four corners of the frame.

24. The time (in seconds) required for a bomb to fall to the ground from an airplane, neglecting the effect of air resistance, is $T = \frac{1}{4}\sqrt{H}$, where H is the height of the airplane (in feet) above the earth. Neglecting air resistance, how many seconds would be required for a bomb to fall from a height of 11,700 ft.?

* By the Pythagorean relation, the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

QUADRATIC EQUATIONS IN ONE UNKNOWN

Two methods for solving quadratic and higher degree equations in one variable have been presented. The first of these, the graphical method, is somewhat laborious, especially when accurate results are desired. The other method, solution of equations by factoring, is a trial-and-error process that can be used only in special cases. The purpose of this chapter is to present the most direct analytic methods of solving quadratic equations. These methods are to be used when the equation cannot be solved at sight by factoring, or when they are more convenient than other methods.

78. Incomplete Quadratic Equations. An equation of the form $ax^2 + bx + c = 0$, in which a, b, and c are not zero, is called a *complete* quadratic equation. If either b or c is zero, the equation is called an *incomplete* quadratic. For example, $x^2 - 16 = 0$ and $2x^2 - 4x = 0$ are incomplete quadratic equations, while

$$3x^2 - 7x + 5 = 0 \qquad \text{and} \qquad x^2 - 2x + 3 = 0$$

are complete quadratics.

Example 1. Solve the equation $2x^2 - 4x = 0$
Incomplete quadratic equations of this type can be solved at sight by removing the common monomial factor, which in this case is $2x$. The equation can be written in the factored form $2x(x - 2) = 0$, from which it can be seen at once that $x = 0$ and $x = 2$ are the roots.

Example 2. Solve the equation $3x^2 - 17x = 0$
Solution: Removing the common monomial factor x, one writes

$$x(3x - 17) = 0$$

The roots of this equation are $x = 0$ and $x = \frac{17}{3}$.

Observe that the method of factoring is not a trial-and-error process in the case of an incomplete quadratic.

Example 3. Solve the incomplete quadratic equation

$$3x^2 - 30 = 0$$

Transposing, $\quad 3x^2 = 30 \quad$ or $\quad x^2 = 10$

Taking square roots in each member,

$$x = \pm\sqrt{10} \doteq \pm 3.16$$

and the two roots of the equation are

$$x \doteq 3.16 \qquad \text{and} \qquad x \doteq -3.16$$

NOTE: Since the decimal answer 3.16 is not exactly equal to $\sqrt{10}$, it is best to leave the answer in the radical form unless otherwise directed.

EXERCISES (78)

Solve and check the following equations:

1. $x^2 - 9x = 0$ 2. $2x^2 = 16$ 3. $3x^2 - 18 = 0$
4. $x^3 - 6x^2 = 0$ 5. $6x^2 - 4x = 0$ 6. $12x^2 - 32x = 0$
7. $x^2 + 3x - 7 = 3(x + 3)$ 8. $3x^2 - 16 = 59$
9. $16x = \dfrac{25}{x}$ 10. $6x^2 - 12 = 2x^2 + 24$
11. $\dfrac{x}{2} + \dfrac{2}{x} = x$ 12. $\dfrac{4}{x^2 + 1} = \dfrac{2}{x^2 - 3}$
13. $(x + 3)(x - 3) = 2x^2 - 10$ 14. $\dfrac{2}{x^2 - 6} = \dfrac{5}{2x^2 - 5}$

Solve the following for x:

15. $3x^2 - 5a^2 = x^2 - a^2$ 16. $\dfrac{7a^2}{2} - x^2 = \dfrac{3x^2}{2}$
17. $2(x^2 - 12a^2) = (x + 2a)(x - 2a)$
18. $\dfrac{x^2 - 4a^2}{9a^2} = \dfrac{x^2 - 9a^2}{4a^2}$

79. Completing the Square. The roots (+3 and −3) of the equation $x^2 = 9$ can be obtained by extracting the square root of each side of the equation. The roots of the equation $(x + 3)^2 = 25$ can be found in the same way. Extracting square roots on each side,

$$x + 3 = \pm 5 \qquad \text{or} \qquad x = \pm 5 - 3$$

Then the roots are

$$x = 5 - 3 = 2 \qquad \text{and} \qquad x = -5 - 3 = -8$$

or

$$x = 2 \qquad \text{and} \qquad x = -8$$

It has been shown that the two roots of the equation $(x + 3)^2 = 25$

are $x = 2$ and $x = -8$. Observe that $(x + 3)^2 = 25$ is equivalent to

$$x^2 + 6x + 9 = 25 \qquad \text{or} \qquad x^2 + 6x = 16$$

Now let us consider how we might solve the equation $x^2 + 6x = 16$, which we know in advance has the roots $x = 2$ and $x = -8$. By adding 9 to each side, we can put this equation in the form

$$x^2 + 6x + 9 = 25 \qquad \text{or} \qquad (x + 3)^2 = 25$$

Observe that this changes the left member $x^2 + 6x$ into the *perfect square* $x^2 + 6x + 9$, making it possible to extract square roots and solve the equation. The amount added to each side is the square of *half* the coefficient of x, *i.e.*, $(\frac{6}{2})^2 = 3^2 = 9$.

Example 1. Solve the equation $x^2 - 18x - 7 = 0$

Solution: Transposing (-7), $x^2 - 18x = 7$

Half the coefficient of x is $\frac{18}{2}$, or 9; hence we add 81 to each side, obtaining

$$x^2 - 18x + 81 = 7 + 81 \qquad \text{or} \qquad x^2 - 18x + 81 = 88$$

But $\qquad x^2 - 18x + 81 = (x - 9)^2$

Then $\qquad (x - 9)^2 = 88$

Extracting square roots,

$$x - 9 = \pm\sqrt{88} \qquad \text{or} \qquad x = 9 \pm \sqrt{88}$$

The roots are $\qquad x = 9 + \sqrt{88} \qquad$ and $\qquad 9 - \sqrt{88}$

From the square root table, $\sqrt{88} \doteq 9.381$

Then $x \doteq 9 + 9.381 = 18.381$, and also $x \doteq 9 - 9.381 = -.381$; so the two roots are $x \doteq 18.381$ and $x \doteq -.381$.

The method used in Example 1 is applicable only when the coefficient of x^2 is 1. For this reason, the equation should first be divided through by the coefficient of x^2, if it is not originally 1.

Example 2. Solve $\qquad 2x^2 - 12x + 8 = 0$

Solution: D(2), $\qquad x^2 - 6x + 4 = 0$

T(4), $\qquad x^2 - 6x = -4$

This equation is now in suitable form for *completing the square* in the left member. Half the coefficient of x is $\frac{6}{2}$, or 3; hence we add 9 to each side, obtaining

$$x^2 - 6x + 9 = 5$$

or $\qquad (x - 3)^2 = 5$

Extracting square roots,

$$x - 3 = \pm\sqrt{5} \quad \text{or} \quad x = 3 \pm \sqrt{5}$$

Then $\qquad x = 3 + \sqrt{5} \quad \text{and} \quad x = 3 - \sqrt{5}$

But $\sqrt{5} \doteq 2.236$; hence $3 + \sqrt{5} \doteq 5.236$ and $3 - \sqrt{5} \doteq .764$.
The roots are $x \doteq 5.236$ and $x \doteq .764$.

The steps in solving quadratic equations by the method of completing the square are as follows:

1. Simplify the equation, if necessary, and write it with the constant term *alone* on the right side of the equation.
2. Divide through the equation by the coefficient of x^2.
3. Square *half* the coefficient of x, and add the result to both sides of the equation.
4. Write the equation as an incomplete quadratic. (This step may be omitted after the procedure becomes familiar.)
5. Extract square roots on each side of the equation, prefixing the sign $\pm$ to the right member.
6. Solve the result of (5) for x, remembering that the sign indicates two values for x.

Fractions will often be encountered:

Example 3. Solve $\qquad 5x^2 - 7x - 4 = 0$

T(4), $\qquad 5x^2 - 7x = 4$

D(5), $\qquad x^2 - \frac{7}{5}x = \frac{4}{5}$

Adding $(\frac{7}{10})^2$, $\qquad x^2 - \frac{7}{5}x + \frac{49}{100} = \frac{4}{5} + \frac{49}{100}$

or $\qquad (x - \frac{7}{10})^2 = \frac{80}{100} + \frac{49}{100} = \frac{129}{100}$

Then $\qquad x - \frac{7}{10} = \pm\frac{1}{10}\sqrt{129} \doteq \pm\frac{1}{10}(11.36) = \pm 1.136$

and $\qquad x \doteq .7 \pm 1.136$

The roots are 1.836 and $-.436$, approximately.

In general, because square roots expressed in decimal form are only approximate, it is best to leave the roots in radical form unless a numerical answer is needed. In the preceding example, it is sufficient to write $x = \frac{7}{10} + \frac{1}{10}\sqrt{129}$ and $x = \frac{7}{10} - \frac{1}{10}\sqrt{129}$, or $x = \dfrac{7 + \sqrt{129}}{10}$ and $x = \dfrac{7 - \sqrt{129}}{10}$.

EXERCISES (79)

In each of the following expressions, what constant term will make the trinomial a perfect square?

1. $x^2 + 6x + ?$
2. $x^2 - 6x + ?$
3. $x^2 - 4x + ?$
4. $x^2 + 2x + ?$
5. $x^2 + 3x + ?$
6. $x^2 - x + ?$
7. $x^2 + 2ax + ?$
8. $x^2 - 6ax + ?$
9. $x^2 + \frac{3}{4}ax + ?$

Solve the following equations by the method of completing the square:

10. $x^2 + 4x - 12 = 0$
11. $x^2 - 8x + 12 = 0$
12. $x^2 - 6x + 8 = 0$
13. $x^2 + 3x - 4 = 0$
14. $3x^2 + 4x - 4 = 0$
15. $x^2 = 10 + x$
16. $2x^2 - 6x - 9 = 0$
17. $2x^2 - 20x = 41$
18. $12x^2 + 19x - 10 = 0$
19. $8x^2 + 30x = 27$
20. $x^2 - 6ax - 7a^2 = 0$
21. $x^2 + 6ax - 10a^2 = 0$
22. $4x^2 + 12ax + a^2 = 0$
23. $x^2 - ax - 6a^2 = 0$
24. $6x^2 + 11ax + 4a^2 = 0$
25. $4x^2 + 7ax - 15a^2 = 0$

80. The Quadratic Formula. In Chap. 2 it was shown that in situations where problems of a given type occur repeatedly, it is desirable to derive or obtain a *formula* with which such problems can be solved by mere substitution of numbers for letters. Quadratic equations appear repeatedly in practically all branches of engineering and science; hence it is worth while to obtain a *quadratic formula* by using the method of completing the square to solve the general quadratic equation, once and for all, as follows:

$$ax^2 + bx + c = 0$$

$$\text{T}(c), \qquad ax^2 + bx = -c$$

$$\text{D}(a), \qquad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Squaring half of $\frac{b}{a}$ gives $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$. Adding this to both members,

$$x^2 + \left(\frac{b}{a}\right)x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \qquad \text{or} \qquad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

The right member is equivalent to $\frac{b^2 - 4ac}{4a^2}$. Taking square roots,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{or} \qquad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Then
$$\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This is the *quadratic formula*, which should be memorized.

Instead of going through the method of completing the square each time one solves a quadratic equation, he needs only to apply the quadratic formula.

Example 1. Solve: $x^2 + 4x - 12 = 0$

Solution: In this equation, $a = 1, \quad b = 4, \quad c = -12$

Hence $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

becomes $$x = \frac{-4 \pm \sqrt{16 - 4(-12)}}{2} = \frac{-4 \pm \sqrt{64}}{2} = \frac{-4 \pm 8}{2}$$

Then $$x = \frac{4}{2} = 2 \text{ and } x = \frac{-12}{2} = -6$$

Example 2: Solve: $3x^2 - 4x - 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{16 - (4)(3)(-6)}}{6}$$
$$= \frac{+4 \pm \sqrt{16 + 72}}{6} = \frac{+4 \pm \sqrt{88}}{6}$$
$$\doteq \frac{4 \pm 9.38}{6} = \frac{13.38}{6} \text{ and } \frac{-5.38}{6}$$

The two roots are 2.23 and $-.897$, approximately.

EXERCISES (80)

Use the quadratic formula to solve the following:

1. $x^2 - 7x + 12 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 - 7x + 6 = 0$
4. $x^2 - 7x + 10 = 0$
5. $x^2 - 6x + 8 = 0$
6. $3x^2 - 2x - 8 = 0$
7. $2x^2 + 3x - 20 = 0$
8. $3x^2 - 7x + 2 = 0$
9. $6x^2 + x - 15 = 0$
10. $2x^3 + 3x^2 - 10x + 12 = x^2(2x + 1)$
11. $x^2 - ax - 6a^2 = 0$
12. $x^2 - 6ax - 7a^2 = 0$
13. $x^2 - x - 10 = 0$
14. $4x^2 - 8x + 4 = 0$
15. $2x^2 - 6x - 9 = 0$
16. $2x^2 + 5x - 6 = 0$
17. $3x^2 + 9x + 4 = 0$
18. $\dfrac{1}{x} + \dfrac{2}{x + 2} = 1$
19. $\dfrac{x}{x + 1} + \dfrac{x + 1}{x + 2} = 1$
20. $(x + 1)^3 - 11 = x^3$
21. $x^2 - 17 = 0$
22. $2x^2 - 15 = 0$

81. Imaginary Numbers. In solving quadratic equations, one

sometimes encounters the indicated square root of a *negative* number:

Example 1. Solve: $x^2 - 2x + 2 = 0$

Solution: $$x = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

The radical $\sqrt{-4}$ does not equal either 2 or -2, since $2^2 = 4$ and also $(-2)^2 = 4$. In fact, there is no ordinary number whose square is negative; hence we conclude that the square root of a negative number is *imaginary*. For the time being, it will be assumed that any root involving the square root of a negative number is meaningless; though in Chap. 14 the significance of such roots will be made apparent. The term *real number* will be used in referring to numbers that are not imaginary.

REVIEW QUESTIONS

1. Describe the following methods of solving the quadratic equation $x^2 - 5x + 6 = 0$:

(*a*) Graphical, (*b*) factoring, (*c*) completing the square, (*d*) quadratic formula.

2. In each of the following cases, which of the methods of question 1 would be most convenient?

(*a*) $x^2 - 3x + 1 = 0$ (*b*) $x^2 - 7x + 12 = 0$ (*c*) $x^2 + 6x + 8 = 0$
(*d*) $6x^2 - 11x + 3 = 0$ (*e*) $3x^2 + 2x - 1 = 0$ (*f*) $2x^2 + 9x + 1 = 0$

3. Describe two types of incomplete quadratic equations, and tell how they can be solved most conveniently.

4. Derive the quadratic formula by solving the equation

$$ax^2 + bx + c = 0$$

CHAPTER 11

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

Before considering the analytic methods used in connection with quadratic equations in two variables, we shall examine their graphical characteristics. The graphs of quadratic equations describe some of the most fundamental geometric and physical relationships in existence.

82. Graphs of Quadratic Equations. The fundamental types of quadratic equations in two variables are illustrated graphically in Fig. 25. Examine these graphs in some detail, in order to observe their general characteristics; then consider the following statements, which are labeled to correspond with the various graphs shown.

A. The graph of an equation of the form

$$y = ax^2 + b$$

is a *parabola* that is symmetrical with respect to the y axis. If x and y are interchanged in the equation, the parabola is symmetrical with respect to the x axis.

B. The graph of an equation of the form

$$xy = a$$

is an equilateral *hyperbola*.

C. The graph of an equation of the form

$$x^2 + y^2 = a$$

is a *circle* whose center is at the origin and whose radius is $\sqrt{a}$.

D. The graph of an equation of the form

$$ax^2 + by^2 = c$$

is an *ellipse* that is symmetrical with respect to the coordinate axes.

E. The graph of an equation of the form

$$ax^2 - by^2 = c$$

is a hyperbola that is symmetrical with respect to the coordinate axes, as at E in Fig. 25.

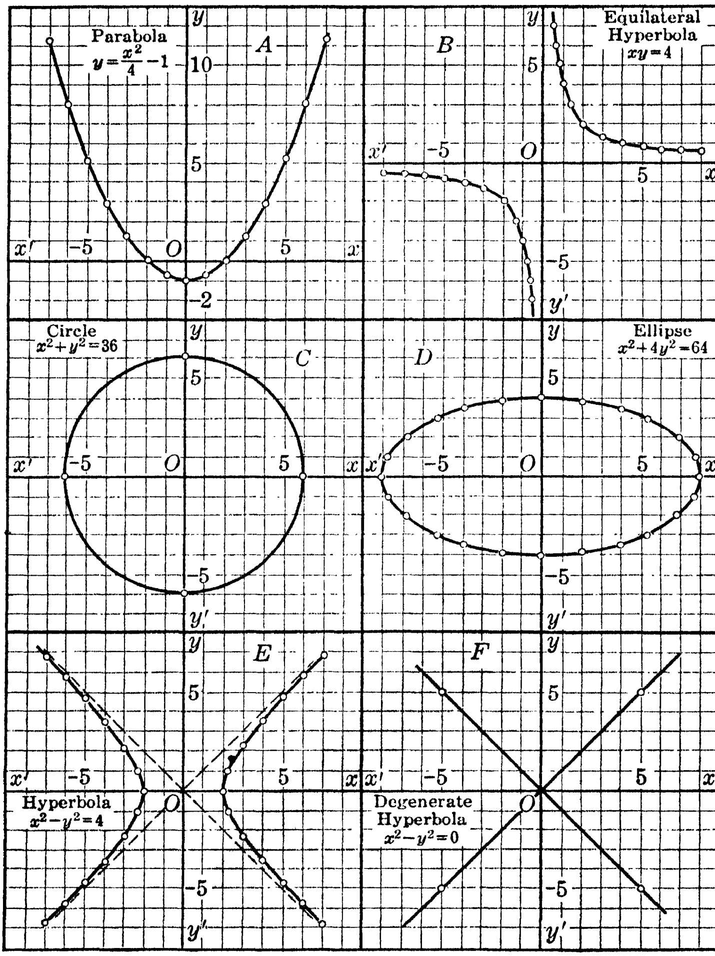


FIG. 25.—Graphs of quadratic equations.

F. The graph of an equation of the form

$$x^2 - y^2 = 0$$

is a *degenerate hyperbola*. Such an equation is called *degenerate* because it degenerates into two linear equations; *viz.*,

$$x^2 - y^2 = 0 \qquad \text{is equivalent to} \qquad (x - y)(x + y) = 0$$

The graph of a quadratic equation in two variables may have any of the following forms: circle, ellipse, parabola, hyperbola, two intersecting lines, two parallel lines, a single line, a single point, or no points at all. In general, these graphs may be in any position with respect to the origin and the coordinate axes. In elementary courses in mathematics, it is customary to consider only those equations whose graphs exhibit some type of symmetry with respect to the origin or the coordinate axes, as in the cases already described. The person who understands the use of

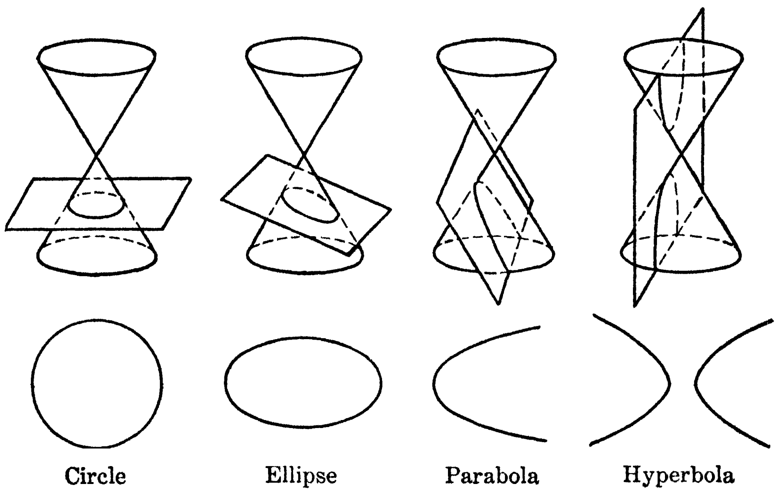


FIG. 26.—Conic sections.

these simpler equations is equipped to handle a great majority of practical problems involving quadratic equations.

The curves obtained by graphing quadratic equations are equivalent to *conic sections*, or sections through a two-napped cone, as shown in Fig. 26. These curves were known to the ancient Greek mathematicians, who regarded them merely as geometric figures. It was not suspected for hundreds of years that any of them except the circle had any applications in the physical universe. Actually, they have countless applications. For example, the orbits of planets (such as the earth) are ellipses, and the path of a comet may be either an ellipse or a parabola. The path of a bullet fired from a gun is also a parabola. The hyperbola has important military applications, such as its use in determining the position of an enemy gun from observations of

the sound made by the gun in firing. The applications of the circle are too numerous to mention.

83. Procedure in Graphing. The method used in plotting the graph of a quadratic equation is the same as that used for linear equations, except that many more points are required to determine the characteristics of the graph.

Example 1. Plot the graph of $xy = 4$.

Solution: Let us solve the equation for y in terms of x and use the result to make up a table of pairs of values for x and y.

From $$xy = 4, \qquad y = \frac{4}{x}$$

If x is positive, y is positive; and if x is negative, y is negative. Since only the *sign* of y is affected by a change in the sign of x, each number in the table can be preceded by the sign $\pm$.

If $x =$	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8
Then $y =$	?	± 4	± 2	$\pm \frac{4}{3}$	± 1	$\pm \frac{4}{5}$	$\pm \frac{2}{3}$	$\pm \frac{4}{7}$	$\pm \frac{1}{2}$

We now plot these points and connect them by a smooth curve, as in the graph at B in Fig. 25.

In the table above, we have chosen only integral values of x. For this reason, we have not yet obtained any points in the region between $x = 0$ and $x = 1$, where the graph is quite steep. In order to obtain points in this range, it is necessary to choose nonintegral values of x. The procedure may be simplified by choosing integral values of y in the desired region and solving for the corresponding values of x. Thus, we write

If $y =$	± 6	7
Then $x =$	$\pm \frac{2}{3}$	$\pm \frac{4}{7}$

and we have the desired points in the range $x = 0$ to $x = 1$. These values can be included in the first table, if we label the rows simply x and y.

It is best to begin each graph by determining its x and y intercepts. In Example 1, however, choosing $x = 0$ produces the result $y = \frac{4}{0}$, which cannot be evaluated. The significance of this

result is that as the value $x = 0$ is approached from either side, the corresponding y distance becomes larger and larger (see B, Fig. 25). We can evaluate it for any value of x except zero, but for $x = 0$ *exactly*, y is large *without limit*, or larger than any actual number. We cannot give it a numerical value or show it on the graph. Such a value is often referred to as *infinite*. The graph of Example 1 shows a similar behavior near $y = 0$, where $x = \frac{4}{0}$.

Example 2. Plot the graph of $x^2 + 4y^2 = 64$, as at D, Fig. 25.
Solution: Solving for y *and* for x, each in terms of the other,

$$y = \tfrac{1}{2}\sqrt{64 - x^2} \qquad \text{and} \qquad x = 2\sqrt{16 - y^2}$$

Now we make up the table, first determining the intercepts:

x	0	± 8	± 2	± 4	± 5.29	± 6.93	± 7.75	± 9	$\pm 2\sqrt{-9}$
y	± 4	0	± 3.87	± 3.46	± 3	± 2	± 1	$\pm \frac{1}{2}\sqrt{-17}$	± 5

If careful choices are made, it is not necessary to calculate a great number of values. For $x = 9$, the value of y involves the square root of a negative number, which is imaginary. Likewise, for $y = 5$, x is imaginary. It can be seen (from the original equation) that y is imaginary for all values of x larger than 8 and that x is imaginary for values of y larger than 4. This limits the graph to the region in which x is between 8 and -8 and y is between 4 and -4 (see D, Fig. 25).

EXERCISES (83)

Carefully plot the graphs of the following equations, *using a full sheet of squared paper for each graph.* You will use these six graphs over and over, both in this chapter and the next; so time spent in making them accurately will be wisely invested. Use the same scale on each graph, choosing the largest convenient scale that will include values from $x = -6$ to $x = 6$ and from $y = -8$ to $y = 8$. Classify each graph by name.

A

1. $y = \dfrac{x^2}{9}$ 2. $y = x^2 - 4x + 2$ 3. $xy = 3$
4. $x^2 + y^2 = 25$ 5. $4x^2 + y^2 = 36$ 6. $x^2 - y^2 = 1$

The following equations are intended as an alternate set to be used instead of the first group, if the instructor so desires.

B

7. $y = \dfrac{x^2}{5}$ 8. $y = x^2 - 2x - 5$ 9. $xy = 5$

10. $x^2 + y^2 = 16$ 11. $2x^2 + y^2 = 25$ 12. $x^2 - y^2 = 2$

84. Systems of Equations, One Quadratic. The graph of a first-degree equation in two unknowns (which is a straight line)

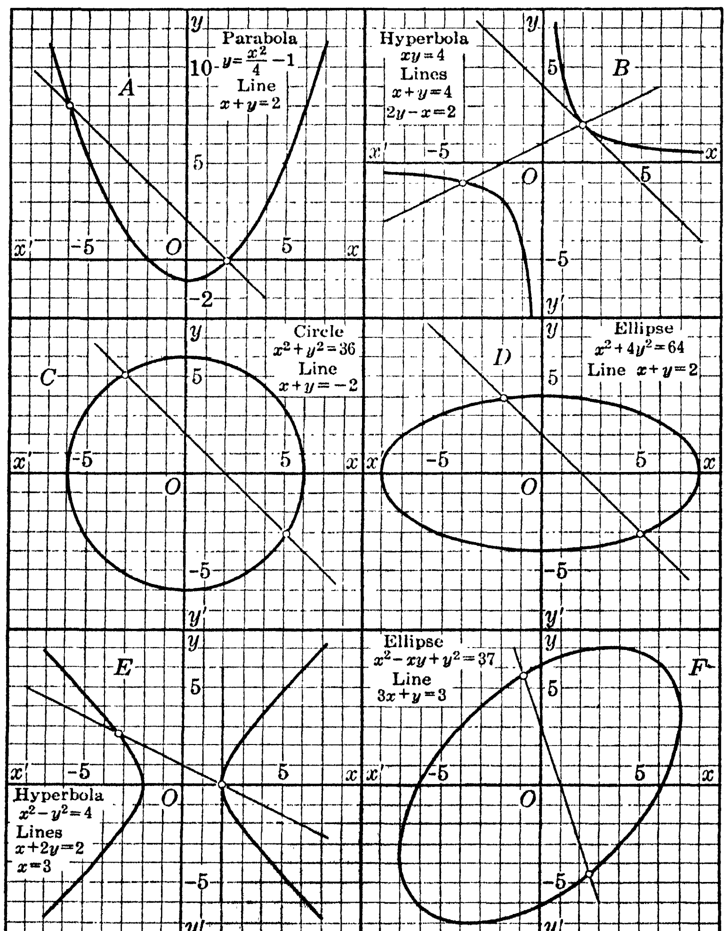


FIG. 27.—Linear-quadratic systems.

may intersect the graph of a quadratic equation in two points, may be tangent to it at a single point, or there may be neither an intersection nor a tangency (Examples are shown in Fig. 27). This means that a pair of equations, one linear and one quadratic,

may have two solutions, one solution, or no solutions; since the coordinates x and y of the points of intersection satisfy both equations and each point of intersection or tangency represents a solution.*

EXERCISES (84)

Record the points of intersection of the graphs in Fig. 27, and check them as (approximate) solutions by substituting the corresponding values of x and y in the equations.

Using the graphs plotted in the preceding section, solve graphically the following systems of equations. In doing this, plot the linear graphs on the same axes with the quadratics, but plot them *very lightly*, since the graphs of the quadratics will be used again. Check your answers to the even problems by substitution.

A

1. $y = \frac{x^2}{9}$, $2y = x + 2$
2. $y = x^2 - 4x + 2$, $x + y = 2$
3. $xy = 3$, $2x - y = 5$
4. $x^2 + y^2 = 25$, $y = 2x - 4$
5. $4x^2 + y^2 = 36$, $2y - x = 2$
6. $x^2 - y^2 = 1$, $x + y = 2$

B

Solve the following (instead of 1 to 6) if you were assigned 7 to 12 in Sec. 83:

7. $y = \frac{x^2}{5}$, $x - y = 0$
8. $y = x^2 - 2x - 5$, $x + y = 2$
9. $xy = 5$, $2x - y = 5$
10. $x^2 + y^2 = 16$, $y = 2x - 4$
11. $2x^2 + y^2 = 25$, $y - x = 2$
12. $x^2 - y^2 = 2$, $x + y = 2$

85. Pairs of Quadratic Equations. The graphs of two quadratic equations may have in common 4 points, 3 points, 2 points, 1 point, or none at all; hence such pairs of equations may have as many as four real solutions or as few as none. Graphical solution of pairs of quadratic equations in two variables is illustrated in Fig. 28.

* These statements refer only to *real* solutions, or solutions that involve only real numbers.

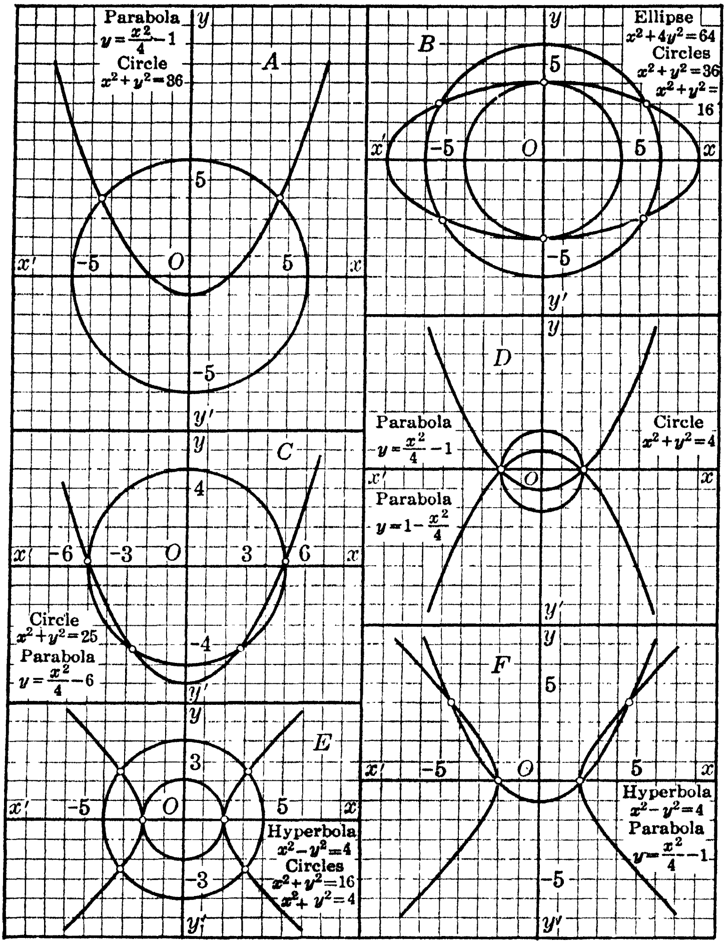


FIG. 28.—Graphs of pairs of quadratic equations.

EXERCISES (85)

Observe the points of intersection of the graphs in Fig. 28, and check them as solutions of the equations.

Using the graphs made in Sec. 83, determine approximate solutions of the following pairs of equations. Do this by transferring a few points from one graph to the other—points that you can see are near the intersections.

A

1. $y = \dfrac{x^2}{9}$, $xy = 3$

2. $x^2 + y^2 = 25$, $xy = 3$

3. $y = \dfrac{x^2}{9}$, $x^2 + y^2 = 25$

4. $y = x^2 - 4x + 2$
$xy = 3$

5. $y = x^2 - 4x + 2$
$x^2 + y^2 = 25$

6. $y = x^2 - 4x + 2$
$y = \frac{x^2}{9}$

7. $x^2 - y^2 = 1$
$x^2 + y^2 = 25$

8. $4x^2 + y^2 = 36$
$y = \frac{x^2}{9}$

9. $y = x^2 - 4x + 2$
$4x^2 + y^2 = 36$

10. $x^2 - y^2 = 1$
$xy = 3$

11. $4x^2 + y^2 = 36$
$x^2 - y^2 = 1$

12. $x^2 - y^2 = 1$
$y = x^2 - 4x + 2$

B

Solve the following systems, if you are using the alternate sets of exercises.

13. $y = \frac{x^2}{5}$
$xy = 5$

14. $x^2 + y^2 = 16$
$xy = 5$

15. $y = \frac{x^2}{5}$
$x^2 + y^2 = 16$

16. $y = x^2 - 2x - 5$
$xy = 5$

17. $y = x^2 - 2x - 5$
$x^2 + y^2 = 16$

18. $y = x^2 - 2x - 5$
$y = \frac{x^2}{5}$

19. $x^2 - y^2 = 2$
$x^2 + y^2 = 16$

20. $2x^2 + y^2 = 25$
$y = \frac{x^2}{5}$

21. $y = x^2 - 2x - 5$
$2x^2 + y^2 = 25$

22. $x^2 - y^2 = 2$
$xy = 5$

23. $2x^2 + y^2 = 25$
$x^2 - y^2 = 2$

24. $x^2 - y^2 = 2$
$y = x^2 - 2x - 5$

CHAPTER 12

SIMPLE SYSTEMS INVOLVING QUADRATICS

The intersections of the graphs of two equations represent real solutions of the equations, or pairs of real values of x and y that satisfy both equations. Conversely, if analytic solution of the equations yields pairs of real values of x and y, they are the coordinates of the points of intersection of the graphs.

86. Linear-quadratic Systems. Any system of equations in two variables in which one equation is linear and the other quadratic can be solved by substitution.

Example 1. Solve the system:

$$y = \frac{x^2}{4} - 1 \qquad (1)$$

$$x + y = 2 \qquad (2)$$

Solution: From (2), $\qquad y = 2 - x$

Substituting in (1), $\qquad 2 - x = \frac{x^2}{4} - 1$

or $\qquad \frac{x^2}{4} + x - 3 = 0$

M(4), $\qquad x^2 + 4x - 12 = 0$

Factoring, $\qquad (x + 6)(x - 2) = 0$

or $\qquad x = -6 \quad \text{and} \quad x = 2$

These values should be substituted into the *linear* equation. Substituting $x = -6$ in (2),

$$-6 + y = 2 \qquad \text{or} \qquad y = 8$$

Substituting $x = 2$ in (2),

$$2 + y = 2 \qquad \text{or} \qquad y = 0$$

Thus, when $x = -6$, $y = 8$; and when $x = 2$, $y = 0$.

When paired correctly, these values represent the coordinates of the points of intersection of the graphs, $(-6, 8)$ and $(2, 0)$. Refer to Fig. **29**, in which the graphs of (1) and (2) are plotted, and check the coordinates of the points of intersection.

Example 2.

$$x^2 + y^2 = 36 \qquad (1)$$
$$x + y = 2 \qquad (2)$$

From (2),
$$y = 2 - x$$
Substituting in (1),
$$x^2 + (2 - x)^2 = 36$$
$$x^2 + 4 - 4x + x^2 = 36$$
$$2x^2 - 4x - 32 = 0$$
D(2),
$$x^2 - 2x - 16 = 0$$

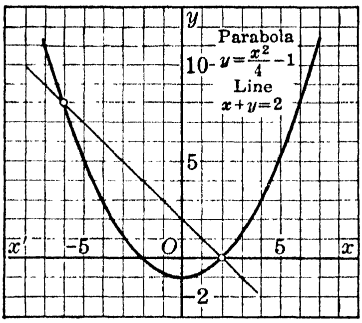


FIG. 29.—Example 1.

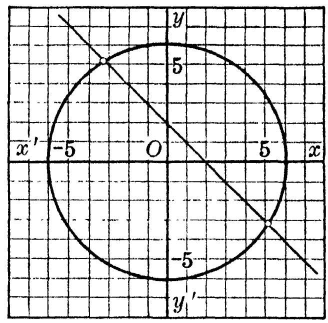


FIG. 30.—Example 2.

Using the quadratic formula,

$$x = \frac{2 \pm \sqrt{4 + 64}}{2} = \frac{2 \pm \sqrt{68}}{2} = 1 \pm \sqrt{17}$$

or
$$x = 1 + \sqrt{17} \qquad \text{and} \qquad x = 1 - \sqrt{17}$$

Substituting $x = 1 + \sqrt{17}$ in (2),
$$1 + \sqrt{17} + y = 2 \qquad \text{or} \qquad y = 1 - \sqrt{17}$$

Substituting $x = 1 - \sqrt{17}$ in (2),
$$1 - \sqrt{17} + y = 2 \qquad \text{or} \qquad y = 1 + \sqrt{17}$$

Pairing the roots, we have the solutions
$$(1 + \sqrt{17}, 1 - \sqrt{17}) \qquad \text{and} \qquad (1 - \sqrt{17}, 1 + \sqrt{17})$$

Now $\sqrt{17} \doteq 4.1$; hence the coordinates of the points of intersection are (5.1, −3.1) and (−3.1, 5.1). Check these values by referring to Fig. 30.

In each of the preceding examples, the numerical value obtained for x was substituted into the *linear* equation, to find y. To substitute into the quadratic equation would produce an extra (incorrect) value for y that would not satisfy the linear equation. The student is not very likely to make such an error, however, since it is simpler to substitute in the linear equation.

EXERCISES (86)

Solve the following systems of equations. Check the answers to those in A (or B) by referring to their graphical solutions, which you obtained in Sec. 84. They are the same systems.

1. $x^2 + y^2 = 25$
   $x + y = 5$
2. $xy = 25$
   $y = 16x$
3. $xy = 10$
   $x = 2y + 1$
4. $x^2 + xy = 10$
   $x - y = 8$
5. $x^2 - y^2 = 7$
   $2x + y = 5$
6. $xy = 4$
   $x = 2y - 2$
7. $\frac{1}{x} + \frac{1}{y} = 4$
   $x + y = 1$
8. $x - 2y = 1$
   $x^2 = 5y^2 + 4$
9. $x^2 - xy + y^2 = 48$
   $y = 2x$
10. $x^2 + xy + y^2 = 7$
    $x - y = 1$
11. $x^2 + y^2 = 5a^2$
    $x - y = a$
12. $x^2 - y^2 = 4a^2$
    $x + y = 2a$

A

13. $y = \frac{x^2}{9}$
    $2y = x + 2$
14. $y = x^2 - 4x + 2$
    $x + y = 2$
15. $xy = 3$
    $2x - y = 5$
16. $x^2 + y^2 = 25$
    $y = 2x - 4$
17. $4x^2 + y^2 = 36$
    $2y - x = 2$
18. $x^2 - y^2 = 1$
    $x + y = 2$

B

19. $y = \frac{x^2}{5}$
    $x - y = 0$
20. $y = x^2 - 2x - 5$
    $x + y = 2$
21. $xy = 5$
    $2x - y = 5$
22. $x^2 + y^2 = 16$
    $y = 2x - 4$
23. $2x^2 + y^2 = 25$
    $y - x = 2$
24. $x^2 - y^2 = 2$
    $x + y = 2$

87. Pairs of Equations, Both Quadratic. In most cases, a pair of quadratic equations in two variables leads to a fourth-degree equation when one of the variables is eliminated.

Example 1.
$$x^2 + y^2 = 25 \qquad (1)$$
$$xy = 5 \qquad (2)$$

From (2),
$$y = \frac{5}{x}$$

Substituting in (1),
$$x^2 + \left(\frac{5}{x}\right)^2 = 25$$
$$x^2 + \frac{25}{x^2} = 25$$

$M(x^2)$,
$$x^4 + 25 = 25x^2$$

$T(25x^2)$,
$$x^4 - 25x^2 + 25 = 0$$

This is a fourth-degree equation. We shall consider methods of solving such equations in the next chapter.

When one of the variables can be eliminated by addition or subtraction, a pair of quadratic equations reduces to a single quadratic equation in one unknown, which can be solved by the methods already learned.

Example 2.
$$x^2 + y^2 = 32 \qquad (1)$$
$$x^2 - y^2 = 18 \qquad (2)$$

Adding,
$$2x^2 = 50$$

D(2), $\quad x^2 = 25 \quad$ or $\quad x = \pm 5$

Substituting $x = 5$ in (1),
$$25 + y^2 = 32 \quad \text{or} \quad y^2 = 7$$

so that
$$y = \pm\sqrt{7}$$

Substitution of $x = -5$ in (1) gives the same result $y = \pm\sqrt{7}$. There are four solutions, or points of intersection,
$$(5, \sqrt{7}), \quad (5, -\sqrt{7}), \quad (-5, \sqrt{7}), \quad \text{and} \quad (-5, -\sqrt{7})$$

Check these values by referring to Fig. 31, in which the graphs of these equations are plotted.

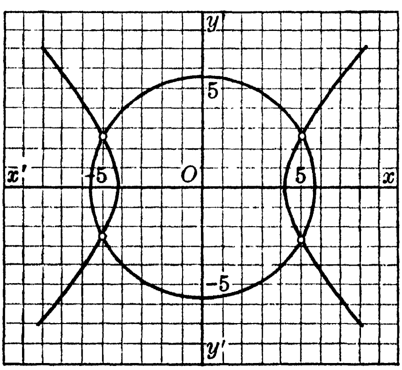


Fig. 31.—Example 2.

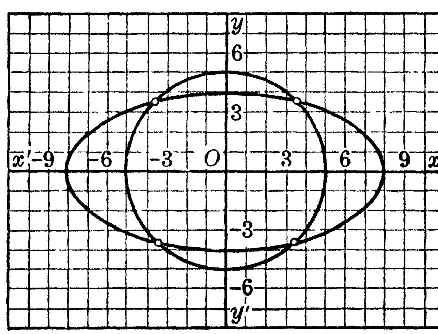


Fig. 32.—Example 3.

Example 3.
$$x^2 + 4y^2 = 64 \qquad (1)$$
$$x^2 + y^2 = 25 \qquad (2)$$

Subtracting,
$$3y^2 = 39$$

D(3), $\quad y^2 = 13, \quad y = \pm\sqrt{13}$

Substituting this value in (2),
$$x^2 + 13 = 25 \quad \text{or} \quad x^2 = 12$$

hence
$$x = \pm\sqrt{12} = \pm 2\sqrt{3}$$

The solutions are $(2\sqrt{3}, \sqrt{13})$, $(2\sqrt{3}, -\sqrt{13})$, $(-2\sqrt{3}, \sqrt{13})$, and $(-2\sqrt{3}, -\sqrt{13})$. Since $\sqrt{3} \doteq 1.73$ and $\sqrt{13} \doteq 3.60$, the points of intersection are

$$(3.46, 3.60), \quad (3.46, -3.60), \quad (-3.46, 3.60), \quad \text{and} \quad (-3.46, -3.60)$$

Check these points by referring to the graphs in Fig. 32, page 167.

EXERCISES (87)

Solve the following systems. Check the solutions for Exercises 2, 4, 6, 8, and 10, by referring to the graphs in Fig. 28, page 162.

1. $x^2 + 7 = 4y^2$
   $x^2 + y^2 = 13$
2. $x^2 - y^2 = 4$
   $y = \frac{x^2}{4} - 1$
3. $x^2 + 4y^2 = 25$
   $4x^2 + y^2 = 40$
4. $x^2 + y^2 = 16$
   $x^2 - y^2 = 4$
5. $4x^2 - 5y^2 = 19$
   $8y^2 - 5x^2 = 20$
6. $x^2 + 4y^2 = 64$
   $x^2 + y^2 = 36$
7. $2x^2 + y^2 = 27$
   $x^2 + 2y^2 = 27$
8. $y = \frac{x^2}{4} - 6$
   $x^2 + y^2 = 25$
9. $2y^2 - 3x^2 = 20$
   $y^2 - x^2 = 11$
10. $x^2 - y^2 = 4$
    $x^2 + y^2 = 4$
11. $2x^2 + y^2 = 18$
    $x^2 + 3y^2 = 15$
12. $x^2 - y^2 = 1$
    $x^2 + y^2 = 31$
13. $x^2 + 3y^2 = 52$
    $x^2 + y^2 = 20$
14. $x^2 + 4y^2 = 64$
    $x^2 - y^2 = 9$
15. $x^2 + xy + y^2 = 25$
    $x^2 + xy - y^2 = 9$

88. Stated Problems. A single quadratic equation yields two solutions, as does also a linear-quadratic system; a pair of quadratic equations may yield as many as four solutions. This does not mean that a stated problem which leads to one or more quadratic equations will always have more than one solution, for the conditions of the problem often exclude some of the solutions of the equations.

Example 1. A man is mowing a lawn that measures 60 by 40 ft. In mowing a strip of uniform width around the lawn, he has mowed half the lawn. How wide is the strip?

Translation: It is easier to express the unmowed area than the area mowed. The unmowed area is a rectangle $(60 - 2x)$ ft. long and $(40 - 2x)$ ft. wide. Then, since the total area of the lawn is

$$60 \times 40 = 2{,}400 \text{ sq. ft.},$$

the unmowed area is $(60 - 2x)(40 - 2x) = 1{,}200$ sq. ft. Thus

$$2{,}400 - 200x + 4x^2 = 1{,}200$$

Solution: Rearranging, $4x^2 - 200x + 1{,}200 = 0$

D(4), $\qquad x^2 - 50x + 300 = 0$

Using the quadratic formula (since the factors are not obvious),

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{50 \pm \sqrt{2{,}500 - 1{,}200}}{2}$$

$$= \frac{50 \pm \sqrt{1{,}300}}{2} \doteq \frac{50 \pm 36.06}{2} = 25 \pm 18.03$$

Then $\qquad x \doteq 43.03$ ft. and 6.97 ft.

It can be seen that 43 is not a reasonable answer, since the entire lawn would be mowed before completing a strip 43 ft. wide. The desired answer is $x = 6.97$, or approximately 7. Checking, the unmowed portion of the lawn is

$$60 - 2x = 60 - 14 = 46 \text{ ft. long}$$

and

$$40 - 2x = 40 - 14 = 26 \text{ ft. wide}$$

The unmowed area is thus $46 \times 26 = 1{,}196$ sq. ft., which is nearly half the area of the lawn. If one uses $x = 6.97$, instead of $x = 7$, the conditions of the problem check very accurately.

Do not overlook the fact that $x = 43.03$ also satisfies the equation obtained in this example. This means that the equation "knows" nothing about mowing; it simply describes a *mathematical* relationship. One must apply common sense to the results obtained from equations, choosing only those solutions that satisfy *all* the conditions of the problem, some of which may not be expressed by the equations.

When there are two unknowns in a problem (whether one or both are to be found), it is sometimes difficult to write a single equation in only one unknown. In such cases, each unknown should be given a symbol. Two equations should be written and then solved by eliminating one of the unknowns. This is often true of situations in which one (or both) of the equations is of second degree.

Example 2. An airplane travels 1,200 miles. If its average speed had been 40 m.p.h. greater, the trip would have taken an hour less. Find the speed of the plane.

Solution: There are two unknowns in this problem; though only one, the speed of the plane, is to be determined. The other unknown is the time required for the trip. In problems of this type, it is usually most

convenient to write the equations in such a way as to isolate the unknown that is *not* to be determined; since it can then be eliminated very easily. Let S be the speed of the plane and T the time required for the trip.

Then
$$T = \frac{1{,}200}{S} \tag{1}$$

If the speed were 40 m.p.h. greater $(S + 40)$, the trip would have taken an hour less $(T - 1)$; hence

$$T - 1 = \frac{1{,}200}{S + 40} \tag{2}$$

Subtracting (2) from (1),

$$1 = \frac{1{,}200}{S} - \frac{1{,}200}{S + 40}$$

and T is eliminated. Clearing of fractions, by multiplying by $S(S + 40)$,

$$S(S + 40) = 1{,}200(S + 40) - 1{,}200S$$

Rearranging, $S^2 + 40S = 1{,}200S + 48{,}000 - 1{,}200S$

or $S^2 + 40S - 48{,}000 = 0$

Solving this equation by means of the quadratic formula,

$$S = \frac{-40 \pm \sqrt{1{,}600 + 192{,}000}}{2}$$

Then $S = \dfrac{-40 \pm \sqrt{193{,}600}}{2} = -20 \pm \sqrt{48{,}400} = -20 \pm 220$

and $S = 200$ m.p.h. or -240 m.p.h.

the second being a meaningless answer.

Checking, $T = \dfrac{1{,}200}{200} \equiv 6$ and $T - 1 = \dfrac{1{,}200}{200 + 40} \equiv 5$

Sometimes it is best (or even necessary) to eliminate one unknown by substitution. This is often the case when one equation is linear and the other quadratic.

Example 3. Find the sides of a rectangle whose area is 90 sq. ft. and whose perimeter is 39 ft.

Solution: Let l be the length and w the width of the rectangle. Then the area is

$$lw = 90 \tag{1}$$

and the perimeter is

$$2l + 2w = 39 \tag{2}$$

From (1),
$$w = \frac{90}{l}$$

Substituting $\frac{90}{l}$ for w in (2),

$$2l + 2\left(\frac{90}{l}\right) = 39$$

$$M(l), \quad 2l^2 + 2(90) = 39l,$$

$$2l^2 - 39l + 180 = 0$$

Then $\quad l = \frac{+39 \pm \sqrt{39^2 - 8(180)}}{4} = \frac{39 \pm \sqrt{1{,}521 - 1{,}440}}{4}$

$$l = \frac{39 \pm \sqrt{81}}{4} = \frac{39 \pm 9}{4} = 12 \text{ and } 7.5$$

Substituting $l = 12$ in (1),

$$w = \frac{90}{l} = \frac{90}{12} = 7.5$$

Substituting $l = 7.5$ in (1),

$$w = \frac{90}{l} = \frac{90}{7.5} = 12$$

The latter result shows that an equation does not "know" the difference between length and width. Since the length l is the longer side, $l = 12$ and $w = 7.5$.

PROBLEMS

Solve the following problems, excluding answers that are meaningless:

1. A man bought a number of chickens of equal value for \$6. If each chicken had cost 5 cents less, he could have bought six more. How many did he buy?

2. Find two numbers whose sum is 26 and whose product is 144.

3. Find the dimensions of a rectangle whose area is 55 sq. in., and whose perimeter is 31 in.

4. A man bought some goats for \$100. Had they cost \$1 less (each), he could have bought five more. How many did he buy?

5. The sum of the reciprocals of two consecutive integers is $\frac{17}{72}$. Find the integers.

6. Find two positive numbers whose product is 119 and whose difference is 10.

7. Find two numbers whose sum is 17 and the sum of whose squares is 149.

8. An airplane takes 50 min. (or $\frac{5}{6}$ hr.) longer to fly 1,500 miles against a 50-m.p.h. wind than it does against a 25-m.p.h. wind. What is the air speed of the plane?

9. A clothing dealer paid \$264 for some suits, all of equal value; then he sold all but one of them, receiving \$363. If he made \$11 profit on each suit sold, how many did he buy?

10. Two pumps together can fill a certain reservoir in 20 hr., while one of the pumps alone can fill the reservoir in 9 hr. less than the other one. How long would it take the faster pump to fill the reservoir?

11. Find two numbers whose difference is 8, and the difference of whose squares is 128.

12. The sum of the squares of two positive consecutive integers is 481. What are they?

13. The square of one number is 18 more than the product of it and a second number. If the sum of the two numbers is 16, what are they?

14. If both the length and width of a rectangle were increased by 2 ft., the area would be increased 60 per cent. The width is 60 per cent of the length. Find the dimensions.

15. The sum of the squares of two positive numbers is 32, while the difference of their squares is 18. What are the numbers?

16. Pilot A flies 1,000 miles in 1 hr. less than pilot B, who flies 50 m.p.h. slower. Find the speed of travel of each.

17. An airplane making a scheduled flight of 1,200 miles is delayed 1 hr., after having gone 240 miles. By increasing its speed 32 m.p.h., it was able to arrive on schedule. Determine its scheduled flying speed.

18. Two squares have areas whose difference is 20. Three times the area of one, plus twice the area of the other, equals 510. Find the lengths of their sides.

19. On a certain nonstop trip, the engineer of a train can save 45 minutes (.75 hr.) by increasing his speed to 4 m.p.h. above the scheduled speed. On the other hand, if he decreased his speed to 4 m.p.h. below schedule, he would arrive 50 minutes late. What is the length of the trip, and what is the scheduled speed of the train?

20. In going 600 ft., the front wheels of a tractor make twenty-five more revolutions than the hind wheels. If the circumference of the front wheel were 2 ft. larger, it would make only ten more revolutions in 600 feet than the hind wheels. Find the circumference of the smaller wheel.

21. The area of one square exceeds twice that of another by 56 sq. in. Its perimeter exceeds that of the other by 24 in. Find the sides of the two squares.

REVIEW QUESTIONS

1. Describe the process of solving a pair of equations, one linear and one quadratic. How does this differ from the solution of a pair of linear equations?

2. How does the method of solution of a pair of quadratic equations differ from that of solving a pair of linear equations, if one unknown can be eliminated by addition or subtraction?

CHAPTER 13

SYSTEMS LEADING TO AN EQUATION OF FOURTH DEGREE

In the preceding chapter it was shown that a pair of quadratic equations in two unknowns will reduce to a single quadratic equation in one unknown, if it is possible to eliminate either unknown by addition and subtraction. In such cases the solution can be completed by means of the quadratic formula. A pair of quadratic equations in which neither unknown can be eliminated by addition or subtraction may lead to an equation of the fourth degree (often called a *quartic* equation), when one unknown is eliminated.

Examples: The equations $x^2 + y^2 = 25$ and $xy = 9$ lead to a fourth-degree equation, while $x^2 + y^2 = 25$ and $x^2 - 2y^2 = 16$ do not. In the first case, elimination of one unknown produces a fourth-degree equation; in the second, it produces only a quadratic equation. Check these statements.

89. Fourth-degree Equations. The analytic solution of fourth-degree equations in general is beyond the scope of this text; though the student can solve them graphically. Some types of fourth-degree equations are solvable, however, by the analytic methods already presented. This is true of equations that contain only even powers of the unknown, so that they are of the form

$$ax^4 + bx^2 + c = 0$$

They can be written in the quadratic form $a(x^2)^2 + b(x^2) + c = 0$, and solved for x^2, either by factoring or by means of the quadratic formula.

Example 1. Solve: $x^4 - 5x^2 + 6 = 0$
Factoring, $(x^2 - 2)(x^2 - 3) = 0$

This equation is satisfied by either $x^2 - 2 = 0$ or $x^2 - 3 = 0$.

From $x^2 - 2 = 0$, $x = \pm\sqrt{2}$
From $x^2 - 3 = 0$, $x = \pm\sqrt{3}$

Thus, there are four solutions

$$x = \sqrt{2},\ -\sqrt{2},\ \sqrt{3},\ \text{and}\ -\sqrt{3}$$

Example 2. Solve: $3x^4 - 10x^2 + 3 = 0$

Solution: Factoring, $(x^2 - 3)(3x^2 - 1) = 0$

From $x^2 - 3 = 0$, $\quad x = \pm\sqrt{3}$
From $3x^2 - 1 = 0$, $\quad x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{3}\sqrt{3}$

The four solutions are

$$x = \sqrt{3},\ -\sqrt{3},\ \tfrac{1}{3}\sqrt{3},\ -\tfrac{1}{3}\sqrt{3}$$

Example 3. Solve: $x^4 - 4x^2 + 2 = 0$

Solution: Since this equation is not factorable at sight, it is necessary to use the quadratic formula:

$$x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2}$$

or $\quad x^2 = 2 \pm \sqrt{2} \doteq 2 \pm 1.414$

Then $\quad x^2 = 3.414$ and $.586$, approximately

From $x^2 \doteq 3.414$, $\quad x \doteq \pm\sqrt{3.414} \doteq \pm 1.848$
From $x^2 \doteq .586$, $\quad x \doteq \pm\sqrt{.586} \doteq \pm .765$

Thus $x \doteq 1.848,\ -1.848,\ .765$, and $-.765$.

EXERCISES (89)

Solve the following:

1. $x^4 - 13x^2 + 36 = 0$
2. $x^4 - 25x^2 + 144 = 0$
3. $2x^4 - 3x^2 + 1 = 0$
4. $2x^4 - 5x^2 + 2 = 0$
5. $x^4 - 5x^2 + 5 = 0$
6. $2x^4 - 6x^2 + 2 = 0$
7. $3x^4 - 8x^2 + 3 = 0$
8. $2x^4 - 7x^2 + 5 = 0$
9. $x^4 - 6m^2x^2 + 9m^4 = 0$
10. $2x^4 - 3n^4x^2 + n^8 = 0$

90. Elimination by Substitution. When one of a pair of quadratic equations can conveniently be solved for one unknown in terms of the other, substitution in the other equation usually leads at once to a fourth-degree equation in one unknown. If this equation contains only even powers of the unknown, it can be solved by the method of the preceding section.

Example 1. Solve the system

$$x^2 + y^2 = 20 \qquad (1)$$
$$xy = 8 \qquad (2)$$

Solution: From (2), $\quad y = \dfrac{8}{x}$

Substituting in (1), $$x^2 + \left(\frac{8}{x}\right)^2 = 20$$

or $$x^2 + \frac{64}{x^2} = 20$$

M(x^2) $$x^4 + 64 = 20x^2$$

or $$x^4 - 20x^2 + 64 = 0$$

Factoring, $$(x^2 - 16)(x^2 - 4) = 0$$

$$x^2 = 16 \quad \text{and} \quad x^2 = 4$$

$$x = \pm 4 \qquad\qquad x = \pm 2$$

Substituting these values in (2),

if $x = 4, \quad -4, \quad 2, \quad -2$

then $y = 2, \quad -2, \quad 4, \quad -4$

The four solutions are (4, 2), (−4, −2), (2, 4), (−2, −4). Check these values by referring to the graphs in Fig. 33.**

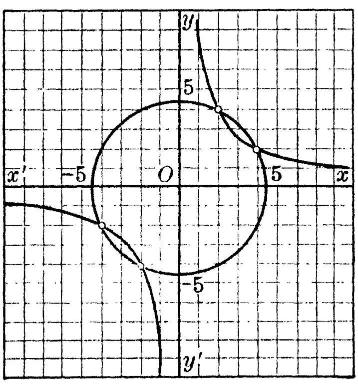


FIG. 33.—Example 1.

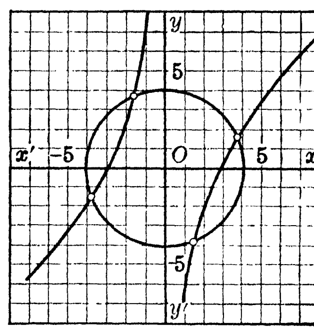


FIG. 34.—Example 2.

Example 2. Solve the system

$$x^2 - xy = 8 \tag{1}$$
$$x^2 + y^2 = 16 \tag{2}$$

Solution: From (1),

$$y = \frac{x^2 - 8}{x}$$

Substituting in (2),

$$x^2 + \frac{(x^2 - 8)^2}{x^2} = 16$$

$$x^2 + \frac{x^4 - 16x^2 + 64}{x^2} = 16$$

M(x^2), $$x^4 + x^4 - 16x^2 + 64 = 16x^2$$

$$2x^4 - 32x^2 + 64 = 0$$

D(2), $$x^4 - 16x^2 + 32 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 128}}{2} = \frac{16 \pm \sqrt{128}}{2} \doteq \frac{16 \pm 11.3}{2}$$

$$x^2 \doteq 13.65 \quad \text{and} \quad x^2 \doteq 2.35$$

From $x^2 \doteq 13.65$, $\quad x^2 \doteq \pm 3.7$
From $x^2 \doteq 2.35$, $\quad x \doteq \pm 1.53$

Substituting these values in (1),

$$\begin{array}{lll} x \doteq 3.7 & 13.65 - 3.7y \doteq 8 & y \doteq 1.53 \\ x \doteq -3.7 & 13.65 + 3.7y \doteq 8 & y \doteq -1.53 \\ x \doteq 1.53 & 2.35 - 1.53y \doteq 8 & y \doteq -3.7 \\ x \doteq -1.53 & 2.35 + 1.53y \doteq 8 & y \doteq 3.7 \end{array}$$

The solutions are (3.7, 1.53), (−3.7, −1.53), (1.53, −3.7), and (−1.53, 3.7). Check these values by referring to Fig. 34, page 175.

CAUTION: Observe that the values of x were substituted in (1), rather than (2). Always substitute numerical values into the equation that will be *of first degree* in the remaining unknown, if it is possible to do so. In this case, substitution in (2) would produce extra (incorrect) values of y, which would not satisfy (1).

Example 3.

$$x^2 - xy = 8 \qquad (1)$$
$$y^2 - xy = -7 \qquad (2)$$

Solution: Solving (1) for y,

$$y = \frac{x^2 - 8}{x}$$

Substituting in (2),

$$\left(\frac{x^2 - 8}{x}\right)^2 - \frac{x(x^2 - 8)}{x} = -7$$

or

$$\frac{x^4 - 16x^2 + 64}{x^2} - x^2 + 8 = -7$$

$\mathrm{M}(x^2)$, $\quad x^4 - 16x^2 + 64 - x^4 + 8x^2 = -7x^2$

This system has reduced to a *quadratic* equation in x.

Collecting, $-x^2 + 64 = 0$, or $x^2 = 64$, so that $x = \pm 8$
Substituting $x = 8$ in (1), $64 - 8y = 8$, or $y = 7$
Substituting $x = -8$ in (1), $64 + 8y = 8$, or $y = -7$

When paired, the solutions are (8, 7) and (−8, −7).

Example 4.

$$y^2 + 3xy = 4 \qquad (1)$$
$$x^2 + 2xy + y^2 = 4 \qquad (2)$$

Solution: In (1), x appears only in first power. Solving (1) for x,

$$x = \frac{4 - y^2}{3y}$$

Substituting in (2),

$$\frac{(4 - y^2)^2}{9y^2} + \frac{2y(4 - y^2)}{3y} + y^2 = 4$$

or

$$\frac{16 - 8y^2 + y^4}{9y^2} + \frac{8 - 2y^2}{3} + y^2 = 4$$

M($9y^2$), $16 - 8y^2 + y^4 + 24y^2 - 6y^4 + 9y^4 = 36y^2$

Collecting, $4y^4 - 20y^2 + 16 = 0$

D(4), $y^4 - 5y^2 + 4 = 0$

Factoring, $(y^2 - 4)(y^2 - 1) = 0$

$$y = \pm 2, \qquad y = \pm 1$$

Substituting these values for y in (1), because it is of first degree in x,

$y = +2$	$4 + 6x = 4$	$x = 0$
$y = -2$	$4 - 6x = 4$	$x = 0$
$y = +1$	$1 + 3x = 4$	$x = 1$
$y = -1$	$1 - 3x = 4$	$x = -1$

and the solutions are $(0, 2)$, $(0, -2)$, $(1, 1)$, and $(-1, -1)$.

EXERCISES (90)

Solve the following systems, which by means of substitution can be reduced to equations of quadratic form in one unknown. Pair the solutions. (NOTE: Most exercises of this type yield solutions involving radicals or imaginary numbers. The following, with a few exceptions, are selected to yield integral, real solutions in order to reduce the amount of labor involved.)

1. $x^2 + y^2 = 2$
   $xy = 1$
2. $xy = 12$
   $x^2 + y^2 = 25$
3. $x^2 + 4y^2 = 16$
   $xy = 4$
4. $x^2 + y^2 = 10$
   $x^2 - xy = 4$
5. $x^2 - y^2 = 16$
   $x^2 + xy = 15$
6. $4y^2 + xy = 8$
   $x^2 + 3xy = 28$
7. $4y^2 - x^2 = 16$
   $x = y^2 - 12$
8. $x^2 + xy = 15a^2$
   $y^2 + xy = 10a^2$
9. $x^2 - y^2 = 5$
   $y^2 + 2x = 6$
10. $3x^2 - y^2 = 5$
    $y = 1 - x^2$

91. Systems Involving a Degenerate Equation. When each equation contains the second powers of both variables, and it is

not possible to eliminate either unknown by addition and subtraction, see if the constant term in one equation is zero. Such an equation can be solved for one variable in terms of the other, by means of the quadratic formula, or possibly even by factoring.

Example 1.

$$7x^2 + 2y^2 = 15 \quad (1)$$
$$2x^2 + 3xy - 2y^2 = 0 \quad (2)$$

Solution: Observe that neither unknown can be eliminated by addition or subtraction. Equation (2), however, can be written in the factored form, $(2x - y)(x + 2y) = 0$ and is therefore satisfied by *either*

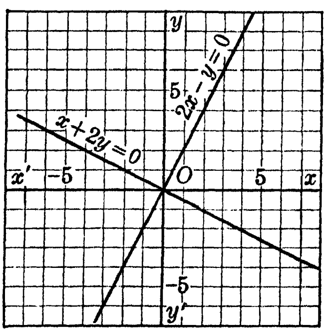


FIG. 35.—Graph of a degenerate equation.

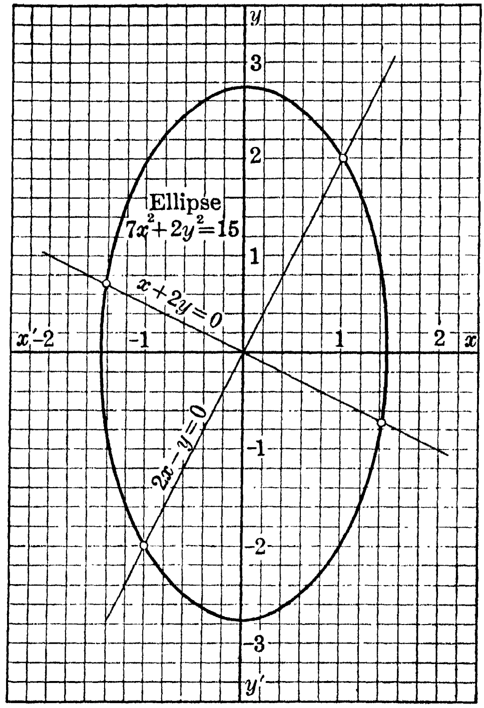


FIG. 36.—Example 1.

$2x - y = 0$ or $x + 2y = 0$. This equation is equivalent to the two linear equations $2x - y = 0$ and $x + 2y = 0$, as shown by its graph in Fig. 35.**

An equation of the form $ax^2 + bxy + cy^2 = 0$ is a *degenerate* quadratic equation whose graph consists of two straight lines intersecting at the origin.* It is referred to as *degenerate* because it "degenerates" into two linear equations upon factoring or application of the quadratic formula.

To solve the system of equations in Example 1, it is necessary only to solve the two simple systems which are obtained by grouping each of the linear equations with (1):

* In some cases the graph is only a single point, the origin.

$7x^2 + 2y^2 = 15$ (1) $2x - y = 0$ (3)	and $7x^2 + 2y^2 = 15$ (1) $x + 2y = 0$ (4)

From (3),

$$y = 2x$$

Substituting in (1),

$$7x^2 + 2(2x)^2 = 15$$

Collecting, $15x^2 = 15$

or $x^2 = 1$, so that $x = \pm 1$.

Substituting in (3),

$$y = 2x = \pm 2$$

Pairing the values, the two solutions are $(1, 2)$ and $(-1, -2)$.

From (4),

$$x = -2y$$

Substituting in (1),

$$7(-2y)^2 + 2y^2 = 15$$

Collecting, $30y^2 = 15$

or $y^2 = \frac{1}{2}$, so that $y = \pm\frac{1}{2}\sqrt{2}$.

Substituting in (4),

$$x = -2(\pm\tfrac{1}{2}\sqrt{2}) = \mp\sqrt{2}$$

The two solutions are $(\sqrt{2}, -\frac{1}{2}\sqrt{2})$, $(-\sqrt{2}, \frac{1}{2}\sqrt{2})$.

In all, there are four solutions:

$$(1, 2), \quad (-1, -2), \quad (\sqrt{2}, -\tfrac{1}{2}\sqrt{2}), \quad \text{and} \quad (-\sqrt{2}, -\tfrac{1}{2}\sqrt{2})$$

Verify these solutions by examining Fig. 36. The method used in this example is often called the *method of reduction to simpler systems*.

It is often necessary to use the quadratic formula in simplifying the degenerate equation; *viz.*,

Example 2.

$$2x^2 + 2xy - y^2 = 0 \qquad (1)$$
$$x^2 + y^2 = 25 \qquad (2)$$

Solution: The expression $2x^2 + 2xy - y^2$ cannot be factored at sight; hence the quadratic formula should be used. Writing the equation as $ax^2 + bx + c = 0$, we see that $a = 2$, $b = 2y$, and $c = -y^2$.

Then $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2y \pm \sqrt{4y^2 + 8y^2}}{4}$$

$$x = \frac{-2y \pm \sqrt{12y^2}}{4} = \frac{-2y \pm 2y\sqrt{3}}{4} = \frac{y}{2}(-1 \pm \sqrt{3})$$

Then $$x = \frac{y}{2}(-1 + \sqrt{3}) \quad \text{and} \quad x = \frac{y}{2}(-1 - \sqrt{3})$$

are the two linear equations equivalent to $2x^2 + 2xy - y^2 = 0$. If we are content to make approximate determinations of the solutions (accurate to, say, 1%), we may write

$$x = \frac{y}{2}(-1 + \sqrt{3}) \doteq \frac{y}{2}(-1 + 1.732) = .366y$$

and $$x = \frac{y}{2}(-1 - \sqrt{3}) \doteq \frac{y}{2}(-1 - 1.732) = -1.366y$$

The two systems to be solved are

$$\boxed{\begin{aligned} x^2 + y^2 &= 25 \qquad (2)\\ x &\doteq .366y \qquad (3)\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} x^2 + y^2 &= 25 \qquad (2)\\ x &\doteq -1.366y \qquad (4)\end{aligned}}$$

Substituting $.366y$ for x in (2),

$$.134y^2 + y^2 \doteq 25$$
$$y^2 \doteq \frac{25}{1.134} \doteq 22.0$$
$$y \doteq \pm\sqrt{22.0} \doteq \pm 4.69$$

Substituting in (3),

$$x = .366(\pm 4.69) \doteq 1.72$$

Pairing values, we have

$$(1.72, 4.69), (-1.72, -4.69).$$

Substituting $-1.366y$ for x in (2),

$$1.87y^2 + y^2 \doteq 25$$
$$y^2 \doteq \frac{25}{2.87} \doteq 8.72$$
$$y \doteq \sqrt{8.72} \doteq \pm 2.96$$

Substituting in (4),

$$x \doteq -1.366(\pm 2.96) \doteq \mp 4.03$$

Pairing values, we have

$$(4.03, -2.96), (-4.03, 2.96).$$

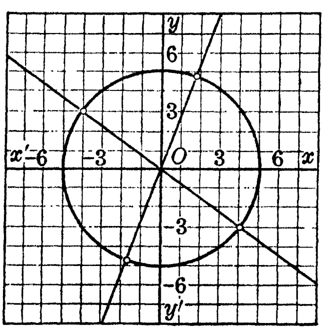


FIG. 37.—Example 2.

The four solutions are (1.72, 4.69), (−1.72, −4.69), (4.03, −2.96), and (−4.03, 2.96). Check these by referring to Fig. 37.

When the two equations contain a common factor, the graph of one of the linear equations forming the degenerate quadratic fails to intersect the graph of the other quadratic equation:

Example 3. $$x^2 - xy - 2y^2 = 0 \qquad (1)$$
$$x^2 - y^2 = 9 \qquad (2)$$

Solution: In factored form, the degenerate equation becomes

$$(x - 2y)(x + y) = 0,$$

so that the systems to be solved are

$$\boxed{\begin{aligned} x^2 - y^2 &= 9 \qquad (2)\\ x - 2y &= 0 \qquad (3)\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} x^2 - y^2 &= 9 \qquad (2)\\ x + y &= 0 \qquad (4)\end{aligned}}$$

From (3), $x = 2y$
Substituting in (2),

$$4y^2 - y^2 = 9$$
$$3y^2 = 9,$$
$$y = \pm\sqrt{3}$$

Substituting in (3),

$$x = 2y = \pm 2\sqrt{3}$$

Pairing values, we have

$$(2\sqrt{3}, \sqrt{3})$$

and

$$(-2\sqrt{3}, -\sqrt{3}).$$

Refer to Fig. 38 for verification of these results.

From (4), $x = -y$
Substituting in (2),

$$y^2 - y^2 = 9, \qquad \text{or} \qquad 0 = 9$$

This is a false equation, indicating that there is no solution for the system comprising (2) and (4).

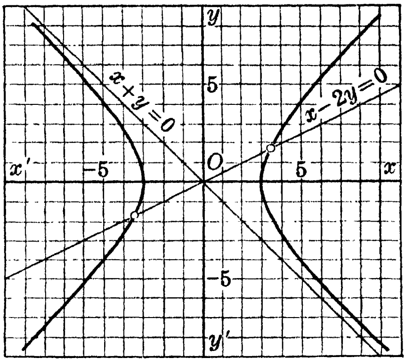


FIG. 38.—Example 3.

EXERCISES (91)

Solve the following by simplifying the degenerate equation. Check your answers to the exercises in A (or B) by plotting the graph of the degenerate equation on the graphs you plotted in Sec. 83.

A

1. $y = \dfrac{x^2}{9}$
   $2x^2 + 3xy - 9y^2 = 0$
2. $y = x^2 - 4x + 2$
   $x^2 - xy - 2y^2 = 0$
3. $xy = 3$
   $4x^2 - 15xy + 9y^2 = 0$
4. $x^2 + y^2 = 25$
   $12x^2 - 7xy - 12y^2 = 0$
5. $4x^2 + y^2 = 36$
   $10x^2 + 3xy - y^2 = 0$
6. $x^2 - y^2 = 1$
   $x^2 - xy - 2y^2 = 0$
7. $4x^2 + y^2 = 36$
   $x^2 - 4xy + y^2 = 0$
8. $y = x^2 - 4x + 2$
   $2x^2 - 6xy + 3y^2 = 0$
9. $x^2 + y^2 = 25$
   $3x^2 - 3xy - y^2 = 0$
10. $xy = 3$
    $x^2 - 6xy + 6y^2 = 0$

B

11. $y = \dfrac{x^2}{5}$
    $2x^2 - 3xy - 5y^2 = 0$
12. $y = x^2 - 2x - 5$
    $9x^2 - 6xy - 8y^2 = 0$
13. $xy = 5$
    $4x^2 - 25xy + 25y^2 = 0$
14. $x^2 + y^2 = 16$
    $6x^2 - 5xy - 6y^2 = 0$

15. $2x^2 + y^2 = 25$
$5x^2 + 4xy - y^2 = 0$

16. $x^2 - y^2 = 2$
$x^2 - xy - 2y^2 = 0$

17. $xy = 5$
$x^2 - 4xy + y^2 = 0$

18. $y = \frac{x^2}{5}$
$x^2 - 2xy - 3y^2 = 0$

19. $x^2 + y^2 = 16$
$x^2 - 4xy - y^2 = 0$

20. $2x^2 + y^2 = 25$
$x^2 - 2xy - 2y^2 = 0$

92. Obtaining a Degenerate Equation. In a system in which the equations contain only terms of second degree, but in neither of which the constant term is zero, a degenerate equation can be obtained by combining the equations in such a way as to *eliminate the constant term,* if the original equations have at least one real solution.

Example 1.

$$x^2 + xy + y^2 = 7 \qquad (1)$$
$$x^2 + y^2 = 10 \qquad (2)$$

Multiplying (1) by 10, and (2) by 7, in order to make the constant terms equal,

$$\begin{array}{rll} 10x^2 + 10xy + 10y^2 = 70 & & (1), \text{ multiplied by } 10 \\ \underline{7x^2 \qquad\quad + \ \ 7y^2 = 70} & & (2), \text{ multiplied by } 7 \\ 3x^2 + 10xy + \ \ 3y^2 = 0 & & \end{array}$$

Subtracting,

This is a degenerate equation, factorable into

$$(3x + y)(x + 3y) = 0$$

The systems to be solved (choosing the simpler of the two original equations) are

$$\boxed{\begin{array}{ll} x^2 + y^2 = 10 & (2) \\ 3x + y = 0 & (4) \end{array}} \quad \text{and} \quad \boxed{\begin{array}{ll} x^2 + y^2 = 10 & (2) \\ x + 3y = 0 & (5) \end{array}}$$

From (4), $y = -3x$

Substituting in (2),

$$x^2 + 9x^2 = 10$$
$$x^2 = 1$$
$$x = \pm 1$$

Substituting in (4),

$$\pm 3 + y = 0$$
$$y = \mp 3$$

Pairing the solutions, we have $(1, -3)$ and $(-1, 3)$.

From (5), $x = -3y$

Substituting in (2),

$$9y^2 + y^2 = 10$$
$$y^2 = 1$$
$$y = \pm 1$$

Substituting in (5),

$$x \pm 3 = 0$$
$$x = \mp 3$$

Pairing the solutions, we have $(-3, 1)$ and $(3, -1)$.

The four solutions are $(1, -3)$, $(-1, 3)$, $(-3, 1)$, and $(3, -1)$.

Carefully examine Fig. 39, in order to observe that *the graphs of the two original equations intersect each other at the same points at which they are intersected by the graph of the degenerate equation obtained from them by eliminating the constant term.* Either of the

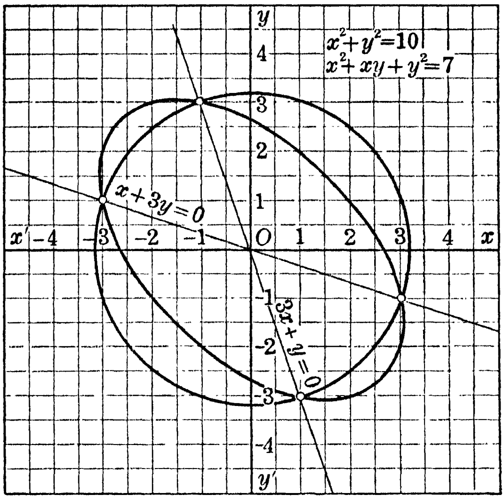


FIG. 39.—Example 1.

original equations, solved with the degenerate equation, will give the desired solutions.

Example 2.

$$14x^2 - 7xy - 10y^2 = 50 \qquad (1)$$
$$x^2 + y^2 = 25 \qquad (2)$$

Multiplying equation (2) by 2 in order to make the constant terms equal,

$$14x^2 - 7xy - 10y^2 = 50 \qquad (1)$$
$$2x^2 + 2y^2 = 50 \qquad (2), \text{ multiplied by } 2$$

Subtracting, $12x^2 - 7xy - 12y^2 = 0$

This is a degenerate equation, factorable into $(3x - 4y)(4x + 3y) = 0$. The systems to be solved (choosing the simpler of the two original equations) are

$$\boxed{\begin{array}{ll} x^2 + y^2 = 25 & (2) \\ 3x - 4y = 0 & (4) \end{array}} \quad \text{and} \quad \boxed{\begin{array}{ll} x^2 + y^2 = 25 & (2) \\ 4x + 3y = 0 & (5) \end{array}}$$

From (4), $x = \dfrac{4y}{3}$ From (5), $x = -\dfrac{3y}{4}$

Substituting in (2),

$$\left(\frac{4y}{3}\right)^2 + y^2 = 25$$

$$\frac{16y^2}{9} + y^2 = 25$$

M(9), $16y^2 + 9y^2 = 225$

$$y^2 = \tfrac{225}{25} = 9$$

$$y = \pm 3$$

Then $x = \dfrac{4y}{3} = 4\left(\dfrac{\pm 3}{3}\right) = \pm 4$

Pairing the solutions, we have (4, 3) and (−4, −3).

Substituting in (2),

$$\left(-\frac{3y}{4}\right)^2 + y^2 = 25$$

$$\frac{9y^2}{16} + y^2 = 25$$

M(16), $9y^2 + 16y^2 = (25)(16)$

$$y^2 = 16$$

$$y = \pm 4$$

Then $x = \dfrac{-3y}{4} = \mp 3$

Pairing the solutions, we have (3, −4) and (−3, 4).

The four solutions are (4, 3), (−4, −3), (3, −4) and (−3, 4). Check these values in Fig. 40.

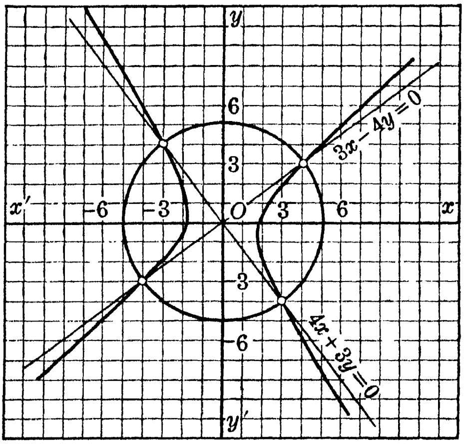


FIG. 40.—Example 2.

EXERCISES (92)

Solve the following:

1. $x^2 + y^2 = 25$
   $13x^2 - 7xy - 11y^2 = 25$
2. $4x^2 + 4xy + y^2 = 4$
   $2x^2 + xy + y^2 = 8$
3. $2x^2 - 3xy + y^2 = 15$
   $x^2 - 2xy + y^2 = 9$
4. $x^2 + xy + y^2 = 63$
   $x^2 - y^2 = -27$
5. $5x^2 - 19xy + 24y^2 = 27$
   $x^2 - 2xy + 3y^2 = 9$
6. $x^2 - y^2 = 16$
   $8x^2 - 9xy - y^2 = 48$
7. $3x^2 + 5xy + 2y^2 = 50$
   $x^2 + 3xy + 2y^2 = 25$
8. $11x^2 - 12xy + 6y^2 = 26$
   $x^2 + y^2 = 13$

93. General Procedure. The general procedure for solving a pair of quadratic equations is as follows. In each case use the first applicable method.

1. Attempt to eliminate one unknown by addition or subtraction. Solve the resulting equation.

2. Solve one of the equations for one unknown in terms of the other, and eliminate by substitution.

3. Look for a degenerate equation (constant term zero). Finding one, use the method of Sec. 91.

4. Failing to find a degenerate equation, obtain one by eliminating the constant term, and proceed as in 3.

5. Whenever any of the preceding methods yields a fourth-degree equation containing odd powers of the variable (*i.e.*, a fourth-degree equation that is not of quadratic form), it can be solved either graphically or by analytic methods taught in more advanced mathematics courses. To solve a fourth-degree equation directly by the graphical method is much more difficult than merely plotting the two quadratic equations and observing their intersections, as was done in Chap. 11; hence the student is advised to use the latter method when this situation is encountered.

EXERCISES (93)

In solving the following, use the general procedure just outlined:

1. $x^2 - 4y^2 = 9$
   $x^2 + y^2 = 39$
2. $x^2 + y^2 = 10$
   $xy = 3$
3. $x^2 - 2y^2 = 9$
   $2x^2 - xy - 3y^2 = 0$
4. $x^2 + y^2 = 16$
   $xy = \sqrt{15}$
5. $x^2 + 2xy + y^2 = 25$
   $x^2 + xy - 2y^2 = 0$
6. $x^2 - xy = 4$
   $x^2 + y^2 = 10$
7. $3x^2 - 2y^2 = 20$
   $x^2 - y^2 = 4.5$
8. $x^2 - y^2 = 7$
   $x^2 - xy = 4$
9. $2x^2 - 3xy + y^2 = 15$
   $x^2 + y^2 = 5$
10. $y^2 - 3xy + 2x^2 = 15$
    $x^2 - 2xy + y^2 = 9$
11. $x^2 + y^2 = 40$
    $xy = 12$
12. $y^2 - x = 14$
    $x^2 - y^2 = 16$
13. $4x^2 + y^2 = 10$
    $2x^2 - xy = 2$
14. $xy - y^2 + 12 = 0$
    $x^2 + xy = 8$
15. $x^2 + y^2 = 25$
    $x^2 - 4y = 20$
16. $x^2 + y^2 = 40$
    $xy = 16$

17. $y^2 + xy + 3x + 1 = 0$
$x + 2y = 3$

18. $5x^2 - 2xy + 4y^2 = 5$
$6x^2 + 10xy - 4y^2 = 10$

19. $7x^2 + 2y^2 = 15$
$x^2 - 2xy + 2y^2 = 5$

20. $2y^2 + xy = 6$
$2xy - y^2 = 2$

REVIEW QUESTIONS

1. Does the system $2x^2 - 3y^2 = 45$, $4x^2 + y^2 = 64$ lead to a fourth-degree equation? Explain.

2. How does one solve a fourth-degree equation in one unknown

   a. If it contains odd powers of the variable as well as even?
   b. If it contains only even powers?

3. When is it convenient to solve a quadratic equation for one variable in terms of the other?

4. When elimination by substitution leads to a fourth-degree equation containing a cubic term, so that it is not of quadratic form, how could you solve it? How would you solve a pair of quadratic equations leading to such a fourth-degree equation?

5. What is the graphical significance of the method of *eliminating the constant term?*

CHAPTER 14

IMAGINARY NUMBERS

There are many equations that have no solutions *in the ordinary sense, i.e.*, we can find no actual numbers that satisfy them. Such equations, however, have been made extremely useful in radio and electrical engineering, and in many other fields, by defining *a new type of number* that makes it possible to solve them and to give a meaning to their solutions.

The numbers used thus far are called *real* numbers in order to distinguish them from numbers of the new type that will be introduced in this chapter. Real numbers include both rational numbers (such as 1, -7, $\frac{3}{11}$) and irrational numbers (such as $\sqrt{2}$, π, $-\sqrt{5}$), either positive or negative.

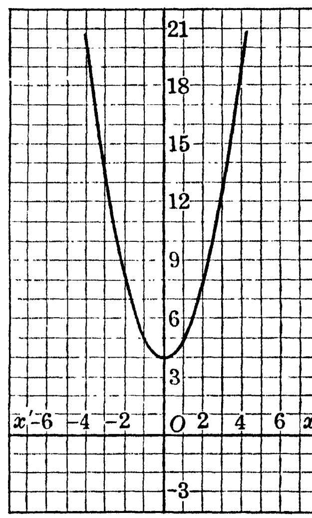


FIG. 41.—Graph of the function $x^2 + 4$.

94. A New Kind of Number. Let us attempt to solve the equation

$$x^2 + 4 = 0$$

$$\text{T(4)}, \qquad x^2 = -4$$

Extracting square root, $\quad x = \pm\sqrt{-4}$

Simplifying, $x = \pm\sqrt{-4} = \pm\sqrt{4}\cdot\sqrt{-1} = \pm 2\sqrt{-1}$

or $\qquad x = 2\sqrt{-1} \quad$ and $\quad -2\sqrt{-1}$

Now, what is the meaning of $\sqrt{-1}$? It is not equal to 1, since $1^2 = +1$. It is not equal to -1, since $(-1)^2 = +1$, also. At this point we are forced to conclude that $\sqrt{-1}$ is meaningless. We can reassure ourselves on this point by referring to the original equation $x^2 + 4 = 0$, or $x^2 = -4$. There is no actual number whose square is negative; hence we cannot expect to obtain a numerical value for x that satisfies this equation. An attempt at graphical solution yields the same information. In Fig. 41, the expression $x^2 + 4$ is plotted as a function of x. Observe that it

does not go to zero; hence the equation $x^2 + 4 = 0$ yields no roots by graphical solution.

Until about 1810, whenever the quantity $\sqrt{-1}$ was encountered in the solution of a problem, it was discarded as meaningless. Near that time, however, several mathematicians began to find ways of interpreting and using it.

Let us represent the quantity $\sqrt{-1}$ by the symbol i, disregarding for the moment the question of what it means. If $i = \sqrt{-1}$, then $i^2 = -1$, since the operation of squaring $\sqrt{-1}$ simply removes the radical sign.

From the equation $x^2 + 4 = 0$, we obtained (above)

$$x = \pm 2\sqrt{-1}$$

which is equivalent to $x = \pm 2i$. We shall now test $x = 2i$ and $x = -2i$ as *roots* of the equation $x^2 + 4 = 0$.

Substituting $x = 2i$, $\quad 4i^2 + 4 = 0$
But $i^2 = -1$ $\quad$ hence $\quad 4i^2 + 4 = -4 + 4 \equiv 0$ (checks)
Substituting $x = -2i$, $\quad 4i^2 + 4 = -4 + 4 \equiv 0$ (checks)

Thus $x = 2i$ and $x = -2i$ do satisfy the equation, although the factor i (and, therefore, the quantities $2i$ and $-2i$) does not represent any actual number. For this reason, we shall refer to i, or $\sqrt{-1}$, as the *imaginary unit* and to the square roots of all negative numbers as *imaginary numbers*.

Any imaginary number can be expressed as the product of a real number and the unit i:

Example 1. $\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3\sqrt{-1} = 3i$
Example 2. $\sqrt{-21} = \sqrt{21} \cdot \sqrt{-1} = i\sqrt{21}$
Example 3. $-\sqrt{-25} = -\sqrt{25} \cdot \sqrt{-1} = -5i$

In simplifying imaginary numbers, it is best to begin by replacing $\sqrt{-1}$ by i:

Example 4. $\sqrt{-16} = i\sqrt{16} = 4i$
Example 5. $-\sqrt{-17} = -i\sqrt{17} \doteq -4.123i$

Higher powers of i can be reduced by replacing i^2 by -1; *viz.*,

$$i^2 = -1, \quad i^3 = i^2 \cdot i = -i, \quad i^4 = (i^2)^2 = (-1)^2 = +1,$$
$$i^5 = (i^2)^2 \cdot i = i, \quad i^6 = (i^2)^3 = -1, \quad i^7 = (i^2)^3 \cdot i = -i, \text{ etc.}$$

EXERCISES (94)

Express each of the following in simplest form, using the imaginary unit:

1. $\sqrt{-49}$	2. $\sqrt{-64}$	3. $-\sqrt{-9}$	4. $-\sqrt{-81}$
5. $\sqrt{-12}$	6. $\sqrt{-18}$	7. $\sqrt{-4a^2}$	8. $-\sqrt{-45}$
9. $-\sqrt{-20a^4}$	10. $2\sqrt{-27x^8}$	11. $\sqrt{-24x^5}$	12. $3\sqrt{-8x^3}$

Solve and check the following:

13. $x^2 + 36 = 0$	14. $2x^2 + 32 = 0$	15. $2x^2 + 90 = 0$
16. $4x^2 + 36 = 0$	17. $3x^2 + 24 = 0$	18. $5x^2 + 50 = 0$

Express in simplest form:

19. i^9	20. i^{10}	21. $(-i)^3$	22. $(-i)^4$
23. $\dfrac{1}{i}$	24. $\dfrac{2}{i^3}$	25. $(-i^3)^3$	26. $\dfrac{1}{-i}$

95. Complex Equations. Even though we refer to numbers like $2i$, $-7i$, etc., as *imaginary* numbers, they have a very real usefulness. Their value lies in the fact that they are "tagged" by the unit i as different from ordinary, or *real* numbers. Because they can be recognized and separated from real numbers, they can be "mixed" in any way with real numbers, then separated at will. This makes it possible to write double, or *complex*, equations, which will be illustrated first by a homely example.

Example 1. Solve for x and y in the equation

$$2x \text{ (apples)} + 3y \text{ (oranges)} = 8 \text{ apples} + 15 \text{ oranges}$$

Solution: At first glance it would appear that we cannot obtain the values of the *two* unknowns x and y from a single equation. However, this equation is really a "double" equation, *because oranges are different from apples.* From this equation we know at once that

$$2x \text{ (apples)} = 8 \text{ (apples)} \qquad \textit{and} \qquad 3y \text{ (oranges)} = 15 \text{ (oranges)}$$

Now we can drop the identifying words, writing

$$2x = 8 \qquad \text{and} \qquad 3y = 15$$
$$\text{D }(2),\ x = 4 \qquad\qquad \text{D }(3),\ y = 5$$

and we have solved for two unknowns.

Consider an equation containing both real and imaginary numbers:

Example 2. $$2x + iy = 17 + 4i$$

Here iy and $4i$ are imaginary numbers and x and 17 are real numbers. Then, because imaginary numbers are different from real numbers, we can write

$$2x = 17 \qquad \text{and} \qquad iy = 4i$$
$$\text{D}(2), \quad x = 8.5 \qquad\qquad \text{D}(i), \quad y = 4$$

After the real and imaginary numbers are separated, the imaginary unit in the relation $iy = 4i$ *drops out,* and the relation between the *real* numbers y and 4 is revealed. Paired with real coefficients, the unit i can "tag along" and preserve a given relationship between those coefficients, no matter what combinations of real numbers appear in the same equation.

> ***In an equation containing both real and imaginary numbers, the equality can be established separately for real and imaginary numbers, yielding two separate equations.***

Example 3. $$2x + ix + 2iy + y = 8 + 7i$$

Solution: Separating real and imaginary numbers,

(Reals)		$2x + y = 8$	(1)
(Imaginaries)		$ix + 2iy = 7i$	(2)
	D(i),	$x + 2y = 7$	(3)
	M(2),	$2x + 4y = 14$	(4)
From (1),		$2x + y = 8$	(1)
Subtracting,		$3y = 6$	
		$y = 2$	

Substituting in (1),

$$2x + 2 = 8 \qquad \text{or} \qquad x = 3$$

Checking by substitution in the original equation,

$$\begin{aligned} 2x + ix + 2iy + y &= 8 + 7i \\ 6 + 3i + 4i + 2 &= 8 + 7i \\ 8 + 7i &\equiv 8 + 7i \end{aligned}$$

Observe that (3), obtained by dividing through (2) by i, involves only *real* numbers.

EXERCISES (95)

Solve the following complex equations:

1. $x + 3ix + 2iy + 4y = 14 + 12i$
2. $2x + 2ix + y - iy - 7 - 5i = 0$
3. $3ix + 4x + 2y - 2iy + 12 + 2i = 0$
4. $ia - ib - 10 - 9i + 5a + 2b = 0$
5. $ix + x - iy - 2y = 2 + 5i$
6. $m + 2mi - 7n - 6ni + 15 + 10i = 0$
7. $ix^2 + y^2 - 5ix + 6i - 3y + 2 = 0$
8. $ix^2 + x + iy^2 - y - 1 = 85i$
9. $2ix^2 + x^2 + 2y^2 - iy^2 + i = 22$
10. $4ix^2 + 4x^2 + 4iy^2 - 3y^2 + 11 = 17i$

96. Equations with Complex Roots. It has been pointed out that some equations have imaginary roots:

Example 1. Solve: $3x^2 + 24 = 0$

Solution: D(3), $x^2 + 8 = 0$

T(8), $x^2 = -8$

$$x = \pm i\sqrt{8} = \pm 2i\sqrt{2}$$

and the roots are $2i\sqrt{2}$ and $-2i\sqrt{2}$.

Checking $x = 2i\sqrt{2}$, $\quad 3(2i\sqrt{2})^2 + 24 = 0$

$$3(8i^2) + 24 = 0 \qquad \text{or} \qquad -24 + 24 \equiv 0$$

The same result is obtained with $x = -2i\sqrt{2}$.

Some equations have roots that contain both a real number and an imaginary number:

Example 2. Solve: $\quad x^2 - 4x + 5 = 0 \quad (1)$

Solution: Using the quadratic formula,

$$x = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

or

$$x = \frac{4 \pm 2i}{2} = 2 \pm i$$

and the roots are $2 + i$ and $2 - i$.

Since an imaginary number always contains i as a *factor*, these roots are not pure imaginary numbers. The sum of a real number and an imaginary number is called a *complex* number, and a root that is a complex number is called a *complex root.*

Checking the complex roots $2 + i$ and $2 - i$, by substitution in (1),

$$(2 + i)^2 - 4(2 + i) + 5 = 0 \qquad (2)$$

but $(2 + i)^2 = (2 + i)(2 + i) = 4 + 4i + i^2 = 4 + 4i - 1 = 3 + 4i$

hence (2) becomes

$$3 + 4i - 4(2 + i) + 5 = 0$$
$$3 + 4i - 8 - 4i + 5 = 0$$
$$0 \equiv 0$$

It is suggested that the student check the remaining root $(x = 2 - i)$ by substitution.

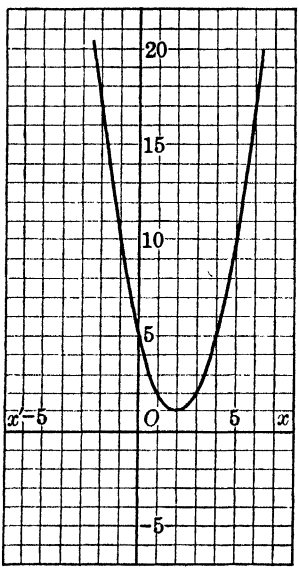


FIG. 42.—Graph of the function $x^2 - 4x + 5$.

The graph of the expression $x^2 - 4x + 5$ as a function of x is shown in Fig. 42. Observe that it has no zeros, thus showing that the equation $x^2 - 4x + 5 = 0$ has no *real* roots.

In multiplying, dividing, adding, and subtracting complex numbers, use the same methods used with radicals, since the unit i represents the radical $\sqrt{-1}$.

Example 3. Evaluate $\dfrac{4 - 3i}{2 + 5i}$.

Solution: Multiplying both numerator and denominator by the conjugate of the denominator,

$$\frac{4 - 5i}{2 + 3i} = \frac{(4 - 5i)(2 - 3i)}{(2 + 3i)(2 - 3i)}$$

Multiplying,

$$\begin{array}{lll} 4 - 5i & \qquad & 2 + 3i \\ \underline{2 - 3i} & & \underline{2 - 3i} \\ 8 - 10i & & 4 + 6i \\ \underline{\quad - 12i + 15i^2} & & \underline{\quad - 6i - 9i^2} \\ 8 - 22i - 15 & & 4 \qquad + 9 \\ -7 - 22i & & 13 \end{array}$$

Thus the result is $\dfrac{-7 - 22i}{13}$.

Example 4. Solve: $\qquad x^2 - 6x + 12 = 0 \qquad (1)$

Solution: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{6 \pm \sqrt{36 - 48}}{2} = \dfrac{6 \pm \sqrt{-12}}{2}$

or $\qquad x = \dfrac{6 \pm i\sqrt{12}}{2} = \dfrac{6 \pm 2i\sqrt{3}}{2} = 3 + i\sqrt{3}$

Checking the root $x = 3 + i\sqrt{3}$ by substitution in (1),

$$(3 + i\sqrt{3})^2 - 6(3 + i\sqrt{3}) + 12 = 0 \qquad (2)$$

Here $(3 + i\sqrt{3})^2 = 9 + 6i\sqrt{3} + 3i^2 = 6 + 6i\sqrt{3}$

Then (2) becomes

$$6 + 6i\sqrt{3} - 18 - 6i\sqrt{3} + 12 \equiv 0, \text{ checks}$$

The student should check the other root $x = 3 - i\sqrt{3}$.

EXERCISES (96)

Evaluate:

1. $(3 - 2i)^2$
2. $(4 - 5i)^2$
3. $(2 - 3i)(3 - 2i)$
4. $(4 - i)^2$
5. $\dfrac{2 - 5i}{5 - 2i}$
6. $\dfrac{3 - 2i}{2 - 3i}$
7. $\dfrac{4 - i}{3 - 2i}$
8. $\dfrac{5 + 7i}{4 + 5i}$
9. $\dfrac{3 - 7i}{14 + 2i}$

Solve the following:

10. $x^2 + 2x + 2 = 0$
11. $x^2 - 6x + 10 = 0$
12. $x^2 - 2x + 17 = 0$
13. $x^2 - 2x + 3 = 0$
14. $x^2 - 5x + 4 = 0$
15. $x^2 - 6x + 9 = 0$
16. $2x^2 - x + 1 = 0$
17. $x^2 - x - 1 = 0$
18. $3x^2 + 7x + 5 = 0$
19. $3x^2 - 6x + 4 = 0$

97. The Discriminant. Consider the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$, which appears under the radical sign, is called the quadratic *discriminant*. By observing whether it is positive, zero, or negative, we can *discriminate* between quadratic equations with real roots and equations with complex or imaginary roots. If $b^2 - 4ac$ is positive, the radical represents the square root of a positive number; hence the roots contain only real numbers.

Examples:

Equation	Discriminant	Roots
$x^2 - 4x + 3 = 0$	$b^2 - 4ac = 16 - 12 = 4$	2 ± 1, or 3 and 1
$2x^2 - 7x + 3 = 0$	$b^2 - 4ac = 49 - 24 = 25$	$\frac{1}{4}(7 \pm 5)$, or 6 and 1
$x^2 - 2x - 7 = 0$	$b^2 - 4ac = 4 + 28 = 32$	$\pm 2\sqrt{2}$
$x^2 - 25 = 0$	$b^2 - 4ac = 0 + 100 = 100$	± 5

In all these cases the roots are *real* and *unequal.*

If $b - 4ac$ is negative (less than zero), the radical represents the square root of a negative number and is therefore an imaginary number. In such cases the roots are, in general, complex numbers.

Examples:

Equation	Discriminant	Roots
$x^2 - 2x + 8 = 0$	$b^2 - 4ac = 4 - 32 = -28$	$1 \pm i\sqrt{7}$
$x^2 - 7x + 13 = 0$	$b^2 - 4ac = 49 - 52 = -3$	$\frac{1}{2}(7 \pm i\sqrt{3})$
$2x^2 - 3x + 2 = 0$	$b^2 - 4ac = 9 - 16 = -7$	$\frac{1}{4}(3 \pm i\sqrt{7})$
$x^2 + 25 = 0$	$b^2 - 4ac = 0 - 100 = -100$	$\pm 5i$

In the last case, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{0 \pm \sqrt{-100}}{2} = \pm 5i$

and the roots are imaginary rather than complex, since they contain no real terms. This equation can of course be solved more conveniently: $x^2 + 25 = 0$, $x^2 = -25$, $x = \pm\sqrt{-25} = \pm 5i$.

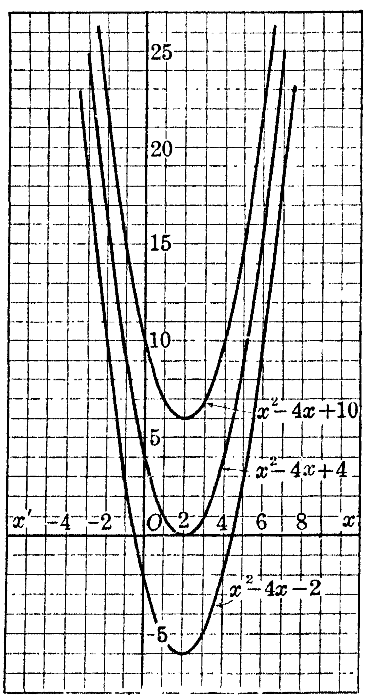


FIG. 43.—Significance of the discriminant.

If the discriminant $b^2 - 4ac$ has the value *zero*, the two roots of the equation are real and *equal*, *i.e.*, there is a *double* root.

Example: $x^2 - 2x + 1 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4}}{2}$$

or the solutions are

$1 + 0$ and $1 - 0$, real and equal

Example: $x^2 + 6x + 9 = 0$

$$x = \frac{-6 \pm \sqrt{36 - 36}}{2}$$

$= -3 + 0$ and $-3 - 0$, real and equal

The significance of the discriminant is also illustrated in Fig. 43, in which the graphs of the functions $x^2 - 4x - 2$, $x^2 - 4x + 4$, and $x^2 - 4x + 10$ are plotted on the same set of axes. Evaluate the discriminant for each of the corresponding equations $x^2 - 4x - 2 = 0$, etc., and observe the following:

1. When the discriminant is positive, the graph indicates two real, unequal roots.

2. When the discriminant is negative, the graph indicates no real roots (and there will be imaginary or complex roots).

3. When the discriminant is zero, the two roots are real and equal; that is, there is only one root, which is double.

EXERCISES (97)

Use the quadratic discriminant to classify the following equations as having real, complex, or imaginary roots. If the roots are real, state whether they are equal or unequal. Solve the first four equations to check their classifications.

1. $x^2 + 4x - 3 = 0$
2. $x^2 - 3x + 3 = 0$
3. $x^2 + 25 = 0$
4. $x^2 - 4x + 4 = 0$
5. $2x^2 + 2x - 5 = 0$
6. $4x^2 - 25 = 0$
7. $mx^2 + 4x + 2 = 0$
8. $a^2x^2 + ax + 1 = 0$
9. $2ax^2 + 3x + 2a = 0$
10. $6n^2x^2 - 5mx - 2m^2 = 0$
11. $3x^2 - 7x + 4 = 0$
12. $11x^2 - 9x + 2 = 0$

State under what conditions the roots of the following equations are real:

13. $x^2 - 5x + m = 0$
14. $2x^2 - 5x + m = 0$
15. $mx^2 - 7x - n = 0$
16. $2x^2 - mx - m = 0$

98. Imaginary Solutions of Systems of Equations. In Fig. 44, the graphs of the equations $x^2 + y^2 = 25$ and $x + y = 5$ are

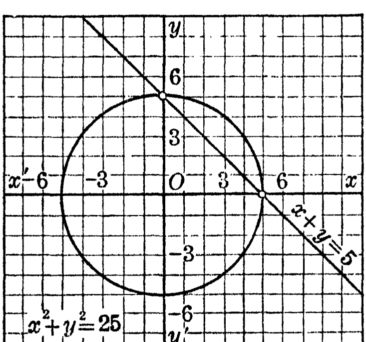


FIG. 44.—Two real solutions.

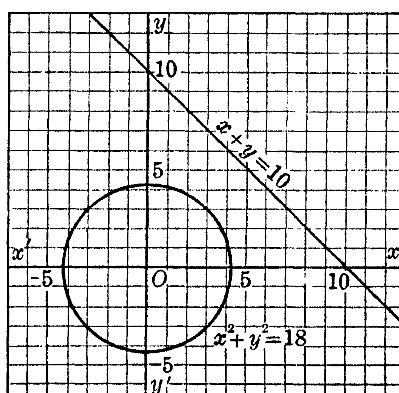


FIG. 45.—No real solutions.

plotted on the same set of axes. It is observed that the graphs intersect in the two points (5, 0) and (0, 5), thus indicating that

there are two real solutions, or pairs of real values of x and y, satisfying both the equations. In Fig. 45, the graphs of $x^2 + y^2 = 18$ and $x + y = 10$ are plotted. These graphs do not intersect; therefore they have no real solution (no pair of real values of x and y that satisfy both equations). If solved by analytic methods, these equations yield *complex roots:*

Example 1.

$$x^2 + y^2 = 18 \qquad (1)$$
$$x + y = 10 \qquad (2)$$

From (2), $y = 10 - x$

Substituting in (1),

$$x^2 + (10 - x)^2 = 18$$

or $x^2 + 100 - 20x + x^2 = 18$

Collecting, $2x^2 - 20x + 82 = 0$

D(2), $x^2 - 10x + 41 = 0$

$$x = \frac{10 \pm \sqrt{100 - 164}}{2} = \frac{10 \pm \sqrt{-64}}{2} = \frac{10 \pm 8i}{2} = 5 \pm 4i$$

Substituting $x = 5 + 4i$ and $x = 5 - 4i$ in (2),

$$x + y = 10 \qquad\qquad x + y = 10$$
$$5 + 4i + y = 10 \qquad\qquad 5 - 4i + y = 10$$
$$y = 5 - 4i \qquad\qquad y = 5 + 4i$$

Pairing the solutions, we have

$$(5 + 4i, 5 - 4i) \quad \text{and} \quad (5 - 4i, 5 + 4i)$$

Checking these values by substituting them in (1),

$$(5 + 4i)^2 + (5 - 4i)^2 = 18$$
$$25 + 40i - 16 + 25 - 40i - 16 \equiv 18, \text{ checks}$$
$$(5 - 4i)^2 + (5 + 4i)^2 \equiv 18, \text{ checks also}$$

Example 2.

$$y = x^2 - 4x + 2 \qquad (1)$$
$$2x + y = 0 \qquad (2)$$

(The graphs of these equations are shown in Fig. 46.)

Solution: From (2), $y = -2x$

Substituting in (1),

$$-2x = x^2 - 4x + 2$$
$$x^2 - 2x + 2 = 0$$
$$x = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Substituting $x = 1 + i$ and $x = 1 - i$ in (2),

$$2(1 + i) + y = 0 \qquad\qquad 2(1 - i) + y = 0$$
$$y = -2 - 2i \qquad\qquad y = -2 + 2i$$

Pairing solutions, we have

$$(1 + i, -2 - 2i) \qquad \text{and} \qquad (1 - i, -2 + 2i)$$

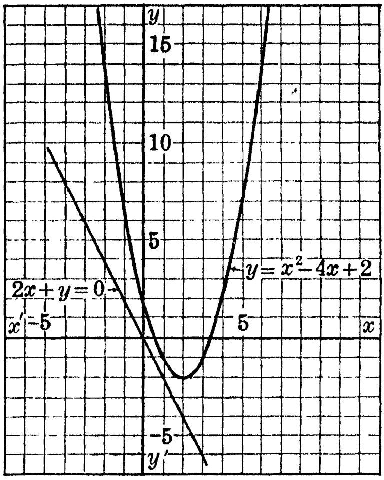


FIG. 46.—Example 2.

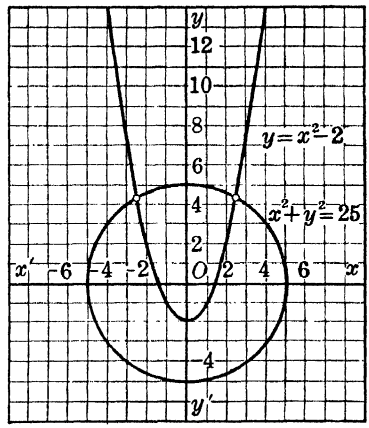


FIG. 47.—Example 3.

Some systems have both real and imaginary solutions:

Example 3.

$$y = x^2 - 2 \tag{1}$$

$$x^2 + y^2 = 25 \tag{2}$$

Eliminating x by addition,

$$y + x^2 + y^2 = x^2 + 23$$

or

$$y^2 + y - 23 = 0$$

Then

$$y = \frac{-1 \pm \sqrt{1 + 92}}{2} = \frac{-1 \pm \sqrt{93}}{2} \doteq \frac{-1 \pm 9.63}{2} \doteq 4.32 \text{ and } -5.32$$

Substituting $y \doteq 4.32$ in (1),

$$4.32 \doteq x^2 - 2$$

Rearranging,

$$x^2 \doteq 6.32$$

Then

$$x \doteq \pm\sqrt{6.32} \doteq \pm 2.52$$

Thus two solutions of the system are

$$(2.52, 4.32) \qquad \text{and} \qquad (-2.52, 4.32), \text{ approximately}$$

Since these solutions involve only real values of x and y, they represent points of intersection of the graphs, as shown in Fig. 47. Check these solutions by referring to the graphs.

The remaining two solutions are obtained by substituting $y \doteq -5.32$ in (1);

$$-5.32 \doteq x^2 - 2$$
$$x^2 \doteq -3.32$$
$$x \doteq \pm i\sqrt{3.32} \doteq \pm 1.82i$$

and the two corresponding solutions are

$$(1.82i, -5.32) \qquad \text{and} \qquad (-1.82i, -5.32)$$

Since the values of x are imaginary, these solutions do not represent points of intersection of the graphs.

In general, a pair of equations will have as many solutions as the product of their respective degrees. Thus a pair of quadratic equations will usually have 2×2 solutions, while a system comprising one quadratic and one linear equation will have 1×2 solutions. Of these, some may be real solutions, denoting intersections of the graphs, and the rest will involve imaginary or complex values of x and y, thus constituting *imaginary solutions*. In the preceding example there are four solutions, two real and two imaginary.

The solutions may be double, as in the following case:

Example 4. $$x^2 + y^2 = 25 \tag{1}$$
$$x + y = \sqrt{50} \tag{2}$$

From (2), $$y = \sqrt{50} - x$$

Substituting in (1),

$$x^2 + (\sqrt{50} - x)^2 = 25$$

or $$x^2 + 50 - 2x\sqrt{50} + x^2 = 25$$

Collecting, $$2x^2 - 2x\sqrt{50} + 25 = 0$$

or $$x = \frac{2\sqrt{50} \pm \sqrt{200 - 200}}{4} = \frac{1}{2}\sqrt{50} \pm 0$$

Observe that there is only one value of x.

Substituting in (2),

$$\tfrac{1}{2}\sqrt{50} + y = \sqrt{50}$$
$$y = \sqrt{50} - \tfrac{1}{2}\sqrt{50} = \tfrac{1}{2}\sqrt{50}$$

In this case there is only one solution, $(\frac{1}{2}\sqrt{50}, \frac{1}{2}\sqrt{50})$, or $(3.54, 3.54)$, called a *double* solution. Refer to Fig. 48, in which the graphs of the two equations are plotted. Observe that there is a point of tangency, rather than the pair of intersections that would be obtained if the line were nearer the center of the circle. A

point of tangency always counts as a double solution; so either a tangency and two intersections, or a tangency and two imaginary

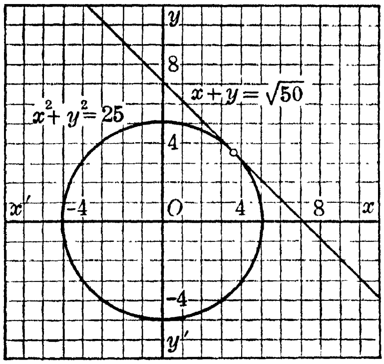


FIG. 48.—Example 4.

solutions, may constitute the four solutions of a pair of quadratic equations.

EXERCISES (98)

Solve the following, stating the number of real solutions in each case:

1. $x^2 + 4y^2 = 64$
   $x^2 + y^2 = 16$
2. $x^2 - y^2 = 16$
   $x^2 + y^2 = 9$
3. $y = \dfrac{x^2}{4} - 1$
   $x^2 + y^2 = 36$
4. $y = x^2 - 4x + 2$
   $2x - y = 11$
5. $xy = 25$
   $y = -x$
6. $x^2 + y^2 = 25$
   $2x^2 + y^2 = 9$
7. $x^2 - y^2 = 4$
   $2x + y = 2$
8. $x^2 + 4y^2 = 64$
   $x^2 - 6x + y^2 = 16$
9. $x^2 + xy = 10$
   $x - y = 8$
10. $4y = x^2 - 4$
    $x^2 + y^2 = 1$
11. $x^2 + y^2 = 16$
    $(x + 1)^2 + 4y^2 = 9$
12. $x^2 + 2xy + y^2 = 49$
    $x - y = 10$
13. $2xy - 9 = y^2$
    $x^2 - y^2 = 0$
14. $3x^2 - y^2 = 12$
    $x^2 + y^2 = 9$
15. $x^2 + y^2 = 4$
    $x = y - 3$
16. $x^2 + y^2 = 4$
    $x + y = 4$
17. $xy = 2$
    $x^2 - 4y^2 = 15$
18. $x^2 - y^2 = 10$
    $2x + y = 1$
19. $2x^2 + 5y^2 = 14$
    $2y = x^2 - 1$
20. $xy = 9$
    $x + y = 2$

99. Extension of the Number Concept. Up to this point, a great deal has been made of the fact that the square root of a negative number is *imaginary*, in order to emphasize that it is different from ordinary positive and negative numbers. It is now time for the student to gain a broader concept of numbers.

Consider the equation $x - 5 = 0$. The value of x which satisfies this equation is 5, a *positive* number. Now consider the equation $x + 5 = 0$. If we were familiar only with the numbers of arithmetic, we would say that this equation has no roots, for there is no arithmetic number x which satisfies this equation. To give this equation a root, it is necessary to define a new kind of number, a negative number, as the result of subtracting a positive number from zero. Thus, " -5 equals zero *minus* 5" defines *what we mean by* a negative number. Before this definition was made, subtracting a positive number from zero was *impossible*.

Now, we cannot eat -5 apples or earn -3 dollars. When first defined, negative numbers were meaningless, but they were soon *given* a significance, and practical uses were found for them. Thus, a negative number became established as opposite to a positive number, representing loss when positive numbers represent profit; or distance below sea level, as opposed to (positive) distance above sea level; etc.

Extracting the square root of a negative number was *impossible* until it was agreed to let i, or $\sqrt{-1}$, represent the *result* of taking the square root of -1. (Remember, subtracting from zero was impossible until a symbol was adopted for the result.) This agreement defines a whole set of numbers, which we *call* imaginary, but which originated the same way as did negative numbers, *i.e.*, *by definition*. No doubt, negative numbers were called imaginary by some people early in their history. As soon as still another kind of number is defined, perhaps it will be called imaginary and our present imaginary numbers will be given a more "real" name.

Imaginary numbers are useful, definite, and practical. Are they any less useful because we cannot count with them? Actually, they are useful because we cannot count with them, or because they are different from the numbers we count with. The student is advised to "take a grain of salt" with the word *imaginary*, regarding it as simply a word of identification for a very definite, useful kind of number.

REVIEW QUESTIONS

1. How many roots has a quadratic equation in one unknown?

2. Can one of the roots of such an equation be real and the other complex or imaginary? Explain.

3. Explain the use of the quadratic discriminant in determining the nature of the roots of such an equation.

4. How many solutions, and of what kind, may there be for a system including one linear and one quadratic equation in two variables?

5. How many solutions, and of what kind, may there be for a pair of quadratic equations in two unknowns?

6. If, in solving a pair of equations, one eliminates y and obtains a quadratic in x, and if the two values of x then obtained are real, does this necessarily mean that the graphs of the equations intersect?

7. If either coordinate of a "point" that represents a solution is complex or imaginary, what sort of solution is it?

8. In solving a pair of quadratic equations, the following solutions are obtained: $(4, -3)$, $(-4, -3)$, $(4, 5i)$, $(-4, -5i)$. In how many points do the graphs intersect?

9. The following are the four solutions of a pair of quadratic equations: $(4, 0)$, $(4, 0)$, $(-4, 3)$, $(-4, -3)$. In how many points do the graphs intersect? What happens at $(4, 0)$? Are there any imaginary solutions?

10. The following are the four solutions of a pair of quadratic equations: $(4, 0)$, $(4, 0)$, $(-\frac{14}{3}, 2.32i)$, $(-\frac{14}{3}, -2.32i)$. How many points of intersection are there? What happens at $(4, 0)$?

CHAPTER 15

EXPONENTS AND ROOTS

In the preceding chapters the student has had occasion to make use of exponents and the laws that govern their use. The extent to which they reduce the labor involved in solving practical problems, however, is not at all obvious at this point and will not be fully seen for some time yet. The student should take it for granted that a thorough understanding of exponents will be of much advantage to him.

100. Laws of Exponents. These rules, which have been used by the student in the preceding chapters, will now be illustrated and proved for the case in which the exponents are positive integers.

Law I. $a^m \cdot a^n = a^{m+n}$ (In multiplying powers of the *same* number or expression, *add* the exponents.)

Examples: $a^3 \cdot a^4 = a^7$, $x^2 \cdot x^3 = x^5$, $2^3 \cdot 2^5 = 2^8$, $(x+y)^2 \cdot (x+y)^3 = (x+y)^5$

Proof: $a^m = a \cdot a \cdot a \cdot a \cdot a \cdots$ (m times)

and $a^n = a \cdot a \cdot a \cdot a \cdots$ (n times)

Since a^m represents m a's multiplied together and a^n represents n a's multiplied together, then $a^m \cdot a^n$ represents $(m+n)a$'s multiplied together; therefore $a^m \cdot a^n = a^{m+n}$.

Law II. $\dfrac{a^m}{a^n} = a^{m-n}$ (In dividing powers of the same number or expression, *subtract* the exponents.)

Examples: $\dfrac{a^5}{a^2} = a^{5-3} = a^2$, $\dfrac{x^8}{x^2} = x^6$, $\dfrac{(x+y)^7}{(x+y)^3} = (x+y)^4$

Proof: Multiply both members of $\dfrac{a^m}{a^n} = a^{m-n}$ by a^n, obtaining

$$a^m = a^{m-n+n}$$

or $a^m \equiv a^m$, which verifies the equality.

Law III. $(a^m)^n = a^{mn}$ In obtaining a power n of a number that already contains an exponent m, multiply the exponent m by the power n.

Examples: $(x^2)^3 = x^{2\cdot 3} = x^6, \qquad (2^2)^3 = 2^{2\cdot 3} = 2^6 = 64$

Proof:
$$\begin{aligned}(a^m)^n &= a^m \cdot a^m \cdot a^m \cdots (n \text{ times})\\ &= a^{m+m+m+m} \cdots (n \text{ times})\\ &= a^{mn}\end{aligned}$$

Law IV. $(ab)^n = a^n b^n$

Examples:
$$(xy)^2 = x^2y^2, \qquad (2y)^3 = 2^3y^3 = 8y^3$$
$$(2ab^3)^4 = 2^4a^4(b^3)^4 = 2^4a^4b^{12} = 16a^4b^{12}$$

Proof:
$$\begin{aligned}(ab)^n &= ab \cdot ab \cdot ab \cdot ab \cdots (n \text{ times})\\ &= a \cdot a \cdot a \cdot a \cdots (n \text{ times}) \cdot b \cdot b \cdot b \cdots (n \text{ times})\\ &= a^n b^n\end{aligned}$$

Law V. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Examples:
$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$
$$\left(\frac{x^3}{y^2}\right)^2 = \frac{x^6}{y^4}, \qquad \left(\frac{2a^3}{3a^2}\right)^3 = \frac{(2a^3)^3}{(3a^2)^3} = \frac{2^3a^9}{3^3a^6} = \frac{8a^3}{27}$$

Proof:
$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots (n \text{ times})\\ &= \frac{a \cdot a \cdot a \cdots (n \text{ times})}{b \cdot b \cdot b \cdots (n \text{ times})}\\ &= \frac{a^n}{b^n}\end{aligned}$$

EXERCISES (100)

Perform the indicated operations:

1. $a^3 \cdot a^5$
2. $x^2 \cdot x^7$
3. $x^2 \cdot y^3 \cdot x^4 \cdot y^5 \cdot z$
4. $\dfrac{x^3y^2z^5}{xy^2z^7}$
5. $(x^2)^3$
6. $(x^3)^2 \cdot \dfrac{12}{2^3}$
7. $(-x)^3 \cdot \dfrac{2^3}{24}$
8. $\dfrac{(-2x)^4}{10^2}$
9. $(-3x^2y^3)^2$
10. $(\frac{2}{5})^3$
11. $\left(\dfrac{y}{x^2}\right)^4$
12. $\dfrac{(2ax^2)^4}{(3ax)^3}$
13. $(xy^2)^2 \cdot (xy^3)^3$
14. $(3x^2y^3)^2 \cdot (z^3y^4)^3$
15. $\dfrac{(3a^2b)^3}{(6ab^2)^3}$
16. $\left(\dfrac{3m^2}{n}\right)^3 \cdot \left(\dfrac{n^2}{3m}\right)^4$

17. $\left(\frac{6z}{y^2}\right)^3\left(\frac{2x}{3z}\right)^2\left(\frac{z^3}{4x}\right)^2$

18. $\frac{(6x^4y^3)^4}{(3xy^2)^3(2x^2y)^5}$

19. $\frac{(2 \cdot 2^3 \cdot 3^2)^4}{(6^3 \cdot 2^2 \cdot 3^3)^2}$

20. $\left(\frac{xn}{m^3}\right)^2\left(\frac{xm^2}{n}\right)^2\left(\frac{m^2n^3}{x}\right)^3$

101. Negative Exponents. In the proof of the second law of exponents $\frac{a^m}{a^n} = a^{m-n}$, it was assumed that m is larger than n. If, however, m is smaller than n; for example, $m = 5$ and $n = 7$, we can write $\frac{a^5}{a^7} = a^{5-7} = a^{-2}$, if it is permissible to use a negative number as an exponent. But $\frac{a^5}{a^7} = \frac{1}{a^2}$; hence $a^{-2} = \frac{1}{a^2}$. Likewise,

$$a^7 \cdot a^{-4} = a^7 \cdot \frac{1}{a^4} = a^3$$

which is the same as would be obtained by applying Law I; *viz.*,

$$a^7 \cdot a^{-4} = a^{7+(-4)} = a^{7-4} = a^3$$

In general, $$a^{-m} = \frac{1}{a^m}$$

The following rule can be applied:

> ***A factor can be moved from the numerator of a fraction to the denominator, or vice versa, if the sign of its exponent is changed.***

NOTE: In applying this rule, it is essential to distinguish *factors* from *terms*.

Examples:

$$\frac{2x^2y}{z} = \frac{2x^2}{zy^{-1}} = \frac{2x^2z^{-1}}{y^{-1}} = \frac{2z^{-1}}{x^{-2}y^{-1}}, \qquad \frac{2ab^{-2}c}{d} = \frac{2ac}{b^2d}$$

Consider the second law of exponents, $\frac{a^m}{a^n} = a^{m-n}$. Let $m = n$, so that $\frac{a^m}{a^m} = a^{m-m} = a^0$. But $\frac{a^m}{a^m} = 1$; hence $a^0 = 1$, and it is seen that *the zero power of any number* (except zero itself) *is equal to unity.*

This result is indicated also by the sequence

$$a^5 \quad a^4 \quad a^3 \quad a^2 \quad a^1 \quad a^0 \quad a^{-1} \quad a^{-2} \quad a^{-3} \quad a^{-4} \quad a^{-5}$$

in which each term equals the preceding term divided by a. Observe that a^0 must equal 1 in order to fit this sequence, since it is between a and $\frac{1}{a}$.

EXERCISES (101)

Express without exponents:

1. 165^0
2. 12^{-1}
3. 3^{-2}
4. $\dfrac{1}{2^{-3}}$
5. $\dfrac{3^2}{3^{-1}}$
6. $(\frac{3}{5})^{-2}$
7. $(\frac{1}{2})^{-3}4^{-2}$
8. $\dfrac{2^0 \cdot 3^{-2} \cdot 5}{5^{-2} \cdot 3^{-3}}$
9. $\dfrac{(\frac{1}{10})^{-3}(\frac{1}{2})^0}{(\frac{2}{3})^{-2}}$
10. 2×10^{-4}

Express in simplest form, using only positive exponents:

11. $x^3 \cdot x^{-1}$
12. $a^5 \cdot a^{-7}$
13. $a^3 \div a^{-2}$
14. $a^0 \cdot a^4$
15. $m^0 \div m^{-3}$
16. $(x^3y^{-1})^{-2}$
17. $\dfrac{x^{-1}y^{-2}}{z^{-3}}$
18. $\dfrac{2^{-3}a^3b^{-4}}{3^{-2}a^{-1}b^{-3}}$
19. $\dfrac{2a^{-5}}{3b^{-2}}$
20. $\dfrac{2^{-3} \cdot 6^{-1}}{12^{-2}}$
21. $\dfrac{(x^2 - y^2)^{-1}}{(x + y)^{-2}}$
22. $\dfrac{x}{m^{-2}} + \dfrac{m}{x^{-2}}$
23. $\dfrac{2m}{n} + \dfrac{3n^{-1}}{m^{-1}}$
24. $x(x + y)^{-1} + y(x - y)^{-1}$
25. $(a^{-1} + b^{-1})(a^{-1} - b^{-1})$
26. $\dfrac{a^2 - b^2}{a^{-1} + b^{-1}}$
27. $\dfrac{(m - n)^{-2}}{(m^{-2} - n^{-2})}$
28. $\dfrac{(x + y)^{-2}}{x^{-2} + y^{-2}}$
29. $x^m \cdot x^n \cdot x^{3-n}$
30. $\dfrac{a^{2n-1}}{a^{n+1}}$
31. $\left(\dfrac{x^a}{x^b}\right)^{-2}$
32. $\dfrac{x^{a+3}y^{2a-1}}{(x^ay^{2-a})^2}$
33. $\dfrac{(x^my^{m-1})^m}{(x^{m-1}y^m)^m}$
34. $\left[\dfrac{(2a^x)^{-1}}{(3a^y)^{-2}}\right]^{-1}$

102. Radicals of Higher Order. The three roots of the equation $x^3 = 8$ are called the *cube roots* of the number 8. One of these roots is 2, since $2 \cdot 2 \cdot 2 = 8$. The other two roots are complex numbers.

The four roots of the equation $x^4 = 81$ are the fourth roots of the number 81. Two of these are 3 and -3; the other two are imaginary numbers.

In general, the n roots of the equation $x^n = a$ are the nth roots of the number a. Of these, the *principal* nth root of a is the *real* root whose sign is the same as the sign of a. A principal root is indicated by the radical sign with the appropriate index, which indicates the order of the radical. Thus, the principal cube root of -8 is $\sqrt[3]{-8} = -2$, and the principal fourth root of 81 is $\sqrt[4]{81} = 3$. The index is omitted in the case of the principal square root. A method for determining the approximate values of higher order principal roots will be presented in Chap. 16.

The radicals used to represent principal roots of higher order can be simplified by the same methods that were used in simplifying quadratic (or square root) radicals, if the significance of the index of the radical is retained throughout.

Example 1. Simplify $\sqrt[3]{54}$.

Solution: The number 54 contains a factor 27 that is a perfect cube.

Thus $$\sqrt[3]{54} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$$

Example 2. Simplify $\sqrt[4]{2^7}$.

Solution: $$\sqrt[4]{2^7} = \sqrt[4]{2^4} \cdot \sqrt[4]{2^3} = 2\sqrt[4]{2^3} = 2\sqrt[4]{8}$$

EXERCISES (102)

Simplify:

1. $\sqrt[3]{\frac{1}{8}}$
2. $\sqrt[3]{\frac{x^3}{27}}$
3. $\sqrt[5]{a^{10}}$
4. $\sqrt{\frac{a^6}{b^6}}$
5. $\sqrt[3]{\frac{m^6}{n^6}}$
6. $\sqrt[3]{16}$
7. $\sqrt[3]{72}$
8. $\sqrt[4]{.0001}$
9. $\sqrt[3]{.008}$
10. $\sqrt[4]{\frac{32x^9}{y^6}}$
11. $\sqrt[3]{32{,}000}$
12. $\sqrt[4]{\frac{81x^{11}}{.0016x^3}}$
13. $\sqrt[3]{.125x^5}$
14. $\sqrt[3]{(x+y)^4}$

103. Fractional Exponents. Consider the expression $a^{\frac{1}{2}}$. Let us assume, without proof, that it is permissible to use a fraction as an exponent and that the laws of exponents are valid for fractional exponents. Accordingly, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 \equiv a$. But this

proves that $(a^{\frac{1}{2}})^2 = a$, or that $a^{\frac{1}{2}}$ is a square root of a. It is customary to define $a^{\frac{1}{2}}$ as the equivalent of $\sqrt{a}$; therefore it represents the *principal* square root of a. Thus, the two square roots of a are $a^{\frac{1}{2}}$ and $-a^{\frac{1}{2}}$, equivalent to $\sqrt{a}$ and $-\sqrt{a}$.

Since $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$, the quantity $a^{\frac{1}{3}}$ has the same meaning as $\sqrt[3]{a}$, the principal cube root of a. Similarly, $a^{\frac{1}{4}} = \sqrt[4]{a}$, or $a^{\frac{1}{n}} = \sqrt[n]{a}$, in general.

Consider the third law of exponents $(a^m)^n = a^{mn}$. If $m = 2$ and $n = \frac{1}{3}$, we have $(a^2)^{\frac{1}{3}} = a^{\frac{2}{3}}$. But $(a^2)^{\frac{1}{3}} = \sqrt[3]{a^2}$, which is the cube root of a^2. Also $a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2$, which is the square of the cube root of a. In general,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

NOTE: When one speaks of "the" cube root or "the" square root, he means the principal root, in each case.

> ***The numerator of a fractional exponent indicates the power to which the base is raised, and the denominator indicates the root that is to be taken.***

Example 1. $5^{\frac{3}{4}} = \sqrt[4]{5^3}$

Example 2. $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

Example 3. $\sqrt[6]{x^3} = x^{\frac{3}{6}} = x^{\frac{1}{2}} = \sqrt{x}$

Example 4. $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$

Or, better, $4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$

Note that the second method involves smaller numbers.

The significance of an exponent in decimal form is easily seen:

Example 5. $3^{0.9} = 3^{\frac{9}{10}} = \sqrt[10]{3^9}$

Example 6. $3^{2.7} = 3^{\frac{27}{10}} = \sqrt[10]{3^{27}}$

or, also, $3^{2.7} = 3^2 \cdot 3^{0.7} = 3^2 \cdot 3^{\frac{7}{10}} = 9\sqrt[10]{3^7}$

Multiplication, division, powers, and roots of quantities having fractional exponents are handled in accordance with the laws of exponents.

Example 7. $x^{\frac{1}{3}} \cdot x^{\frac{1}{6}} = x^{\frac{1}{3}+\frac{1}{6}} = x^{\frac{2}{6}+\frac{1}{6}} = x^{\frac{3}{6}} = x^{\frac{1}{2}}$

Example 8. $2x^{\frac{2}{3}} \cdot x^{\frac{1}{5}} = 2x^{\frac{2}{3}+\frac{1}{5}} = 2x^{\frac{10}{15}+\frac{3}{15}} = 2x^{\frac{13}{15}}$

Example 9. $\dfrac{a^{\frac{1}{3}}}{a^{\frac{1}{4}}} = a^{\frac{1}{3}-\frac{1}{4}} = a^{\frac{4}{12}-\frac{3}{12}} = a^{\frac{1}{12}}$

Example 10. $(a^{\frac{1}{4}})^2 = a^{\frac{2}{4}}$, $(a^{\frac{3}{4}})^3 = a^{\frac{9}{4}} = a^{2\frac{1}{4}} = a^2 \sqrt[4]{a}$

Example 11. $\sqrt{x^{\frac{3}{4}}} = (x^{\frac{3}{4}})^{\frac{1}{2}} = x^{\frac{3}{4}\cdot\frac{1}{2}} = x^{\frac{3}{8}}$

Example 12. $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = \sqrt[3]{64} = 4$

or, better, $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$

Note that the second method involves smaller numbers.

When the exponents are expressed in decimal form, multiplication and division can be done very easily:

Example 13. $x^{2.61} \cdot x^{3.12} = x^{5.73}$

Example 14. $x^{3.48} \div x^{2.15} = x^{1.33}$

Example 15. $\dfrac{x^{3.42} \cdot x^{2.85}}{x^{1.34} \cdot x^{2.31}} = \dfrac{x^{6.27}}{x^{3.65}} = x^{2.62}$

EXERCISES (103)

Express as radicals:

1. $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $x^{\frac{2}{3}}$, $a^{\frac{7}{10}}$, $a^{0.9}$, $x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$
2. $x^{1\frac{1}{2}}$, $x^{\frac{5}{6}}$, $x^{2\frac{1}{3}}$, $(x+y)^{\frac{1}{2}}$, $(x-y)^{-\frac{2}{3}}$
3. $2x^{\frac{1}{2}}$, $(2x)^{\frac{1}{2}}$, $8x^{\frac{1}{3}}$, $(8x)^{\frac{1}{3}}$

Express without radicals:

4. $\sqrt[3]{x}$, $\sqrt[3]{x^2}$, $\sqrt{x^3}$, $\sqrt[5]{a^4}$, $\sqrt[4]{a^3}$
5. $\sqrt[6]{x^2}$, $\sqrt[3]{x^4}$, $\sqrt[2]{x^5}$, $\sqrt{x+y}$, $\sqrt[3]{(x+y)^2}$
6. $x\sqrt{y}$, $\sqrt{mn}$, $a\sqrt[2]{b^3}$, $\sqrt[3]{ab^2}$, $\sqrt[3]{ab^2c^3}$

Evaluate numerically:

7. $36^{\frac{1}{2}}$	8. $27^{\frac{1}{3}}$	9. $16^{\frac{3}{4}}$	10. $8^{\frac{2}{3}}$
11. $4^{\frac{3}{2}}$	12. $8^{-\frac{2}{3}}$	13. $(-8)^{\frac{2}{3}}$	14. $32^{\frac{3}{5}}$
15. $32^{-\frac{3}{5}}$	16. $(\frac{8}{27})^{\frac{2}{3}}$	17. $(\frac{1}{8})^{-\frac{1}{3}}$	18. $(\frac{16}{25})^{-\frac{1}{2}}$
19. $(.008)^{\frac{1}{3}}$	20. $(125)^{\frac{2}{3}}$	21. $(.125)^{\frac{2}{3}}$	22. $(.25)^{-\frac{1}{2}}$

Express with a single exponent:

23. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$	24. $x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$	25. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{4}}$
26. $a \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$	27. $m^{\frac{2}{3}} \cdot m^{\frac{3}{4}}$	28. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{4}}$
29. $a^{\frac{3}{4}} \div a^{\frac{1}{2}}$	30. $m^{\frac{2}{3}} \div m^{\frac{3}{4}}$	31. $m^{\frac{5}{6}} \div m^{\frac{2}{3}}$
32. $2^{\frac{1}{2}} \div 2^{\frac{1}{3}}$	33. $\dfrac{x^{\frac{3}{4}}}{x^{\frac{1}{4}}}$	34. $\dfrac{x^{\frac{2}{3}}}{x^{\frac{3}{4}}}$
35. $x^{1.75} \cdot x^{2.58}$	36. $\dfrac{x^{2.675}}{x^{1.434}}$	37. $\dfrac{10^{2.451} \cdot 10^{3.612}}{10^{4.465}}$

38. $\dfrac{5^{3.43} \cdot 5^{6.21} \cdot 5^{1.11}}{5^{4.42} \cdot 5^{2.63}}$

39. $\dfrac{10^{1.41} \cdot 10^{2.63} \cdot 10^{3.22}}{10^{0.67} \cdot 10^{1.27} \cdot 10^{4.42}}$

40. $(a^{\frac{1}{3}})^2$

41. $(x^{\frac{2}{3}})^6$

42. $(a^{\frac{3}{2}})^{\frac{2}{3}}$

Express in simplest form:

43. $(m^{\frac{1}{3}}n^{\frac{1}{2}})^6$

44. $\left(\dfrac{x^2}{y^3}\right)^{\frac{1}{6}}$

45. $\dfrac{x^{\frac{2}{3}}}{x^{\frac{3}{4}}}$

46. $\left(\dfrac{x^{\frac{2}{3}}}{x^{\frac{3}{4}}}\right)^{-\frac{6}{5}}$

47. $\left(\dfrac{x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}}{x^{\frac{7}{12}}}\right)^3$

48. $\dfrac{(xy^2)^{\frac{1}{3}} \cdot (x^2y)^{\frac{1}{3}}}{y^{\frac{1}{4}}}$

49. $\dfrac{(a^{\frac{3}{2}})^{\frac{1}{2}} \cdot (b^4)^{\frac{1}{4}}}{(b^{\frac{3}{2}})^{\frac{1}{3}}}$

50. $\left(\dfrac{x^{\frac{2}{3}}}{y^2}\right)^{\frac{1}{2}} \cdot \left(\dfrac{y^6}{x}\right)^{\frac{1}{3}}$

51. $\dfrac{x^{\frac{1}{2}}y^{\frac{2}{3}} \cdot x^{\frac{1}{3}}y^{\frac{1}{2}}}{x^{\frac{1}{12}}y^{\frac{5}{12}}}$

52. $\dfrac{(a^{\frac{1}{3}})^3 \cdot (b^{\frac{1}{3}})^3 \cdot c^{\frac{1}{6}}}{(a^{\frac{1}{3}})^{\frac{1}{2}} \cdot (b^{\frac{1}{2}})^3 \cdot (c^{\frac{1}{3}})^{-\frac{1}{2}}}$

53. $\dfrac{10^{2.15} \cdot (10^{3.60})^{\frac{1}{3}} \cdot (10^{3.33})^{\frac{1}{3}}}{(10^{4.60})^{\frac{1}{2}} \cdot 10^{3.25}}$

104. Radicals of Different Orders. Before combining the radicands of two or more radicals by multiplication or division, it is necessary to express them as radicals of the same order:

Example 1. Multiply $\sqrt{2} \cdot \sqrt[3]{3}$.

Solution: $\sqrt{2} = \sqrt[6]{2^3}$ and $\sqrt[3]{3} = \sqrt[6]{3^2}$

Hence $\sqrt{2} \cdot \sqrt[3]{3} = \sqrt[6]{2^3} \cdot \sqrt[6]{3^2} = \sqrt[6]{72}$

These operations can be performed most conveniently by means of fractional exponents, *viz.*,

$$\sqrt{2} \cdot \sqrt[3]{3} = 2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \cdot 3^{\frac{2}{6}} = (2^3)^{\frac{1}{6}} \cdot (3^2)^{\frac{1}{6}} = (72)^{\frac{1}{6}} = \sqrt[6]{72}$$

The recommended procedure consists of expressing the fractional exponents in terms of their lowest common denominator, so that the result can be expressed with a single fractional exponent, or a single radical sign.

Example 2. Multiply $\sqrt[3]{2} \cdot \sqrt[4]{3}$.

Solution: $\sqrt[3]{2} \cdot \sqrt[4]{3} = 2^{\frac{1}{3}} \cdot 3^{\frac{1}{4}} = 2^{\frac{4}{12}} \cdot 3^{\frac{3}{12}} = (2^4 \cdot 3^3)^{\frac{1}{12}}$

$= (16 \times 27)^{\frac{1}{12}} = (432)^{\frac{1}{12}}$ or $\sqrt[12]{432}$

In order to multiply or divide radicals of different orders:

1. Express the radicals in terms of fractional exponents.

2. Express the fractional exponents in terms of their L.C.D.

3. Express the result in terms of a single fractional exponent, or a single radical sign, as desired.

Example 3. Multiply $\sqrt[3]{x^2} \cdot \sqrt[5]{x^3}$.

Solution: $\sqrt[3]{x^2} \cdot \sqrt[5]{x^3} = x^{\frac{2}{3}} \cdot x^{\frac{3}{5}} = x^{\frac{10}{15}} \cdot x^{\frac{9}{15}} = x^{\frac{19}{15}}$

Then $x^{\frac{19}{15}} = x \cdot x^{\frac{4}{15}} = x \sqrt[15]{x^4}$

Example 4. Multiply $\sqrt{x^3} \cdot \sqrt[3]{y^2}$.

Solution:

$$\sqrt{x^3} \cdot \sqrt[3]{y^2} = x^{\frac{3}{2}} \cdot y^{\frac{2}{3}} = x^{\frac{9}{6}} \cdot y^{\frac{4}{6}} = x(x^{\frac{3}{6}}y^{\frac{4}{6}}) = x(x^3y^4)^{\frac{1}{6}} = x \sqrt[6]{x^3y^4}$$

Example 5. Express $\dfrac{\sqrt[3]{4x^5}}{\sqrt{2x^3}}$ with a single radical sign.

Solution: $\dfrac{\sqrt[3]{4x^5}}{\sqrt{2x^3}} = \dfrac{2^{\frac{2}{3}}x^{\frac{5}{3}}}{2^{\frac{1}{2}}x^{\frac{3}{2}}} = 2^{\frac{2}{3}-\frac{1}{2}}x^{\frac{5}{3}-\frac{3}{2}} = 2^{\frac{4}{6}-\frac{3}{6}}x^{\frac{10}{6}-\frac{9}{6}} = 2^{\frac{1}{6}}x^{\frac{1}{6}} = (2x)^{\frac{1}{6}} = \sqrt[6]{2x}$

EXERCISES (104)

Multiply:

1. $\sqrt{3} \cdot \sqrt[3]{2}$
2. $\sqrt[4]{3} \cdot \sqrt[3]{2}$
3. $\sqrt[3]{4} \cdot \sqrt[4]{3}$
4. $\sqrt{3} \cdot \sqrt[4]{5}$
5. $\sqrt[3]{2} \cdot \sqrt[6]{3}$
6. $\sqrt[3]{x^2} \cdot \sqrt[5]{x^3}$
7. $\sqrt[3]{a^2} \cdot \sqrt[7]{a^5}$
8. $\sqrt{xy} \cdot \sqrt[3]{x^2y^2}$
9. $\sqrt[4]{2xy^3} \cdot \sqrt[6]{5x^3y^5}$
10. $\sqrt[6]{m^3n^5} \cdot \sqrt[5]{m^4n^3}$
11. $\sqrt[10]{5x^2y^6} \cdot \sqrt[5]{3x^4y^2}$
12. $\sqrt{2x} \cdot \sqrt[3]{4x^2}$

Divide:

13. $\dfrac{\sqrt{3}}{\sqrt[3]{2}}$
14. $\dfrac{\sqrt[3]{2}}{\sqrt[4]{3}}$
15. $\dfrac{\sqrt[3]{4}}{\sqrt[4]{3}}$
16. $\dfrac{\sqrt{3}}{\sqrt[4]{5}}$
17. $\dfrac{\sqrt[3]{a^2}}{\sqrt[7]{a^5}}$
18. $\dfrac{\sqrt[3]{x^2y^2}}{\sqrt{xy}}$
19. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}}$
20. $\dfrac{\sqrt[3]{a^2b^2}}{\sqrt[4]{a^3b^3}}$
21. $\dfrac{\sqrt[3]{a^2b^5}}{\sqrt[2]{a^3b^5}}$
22. $\dfrac{\sqrt[3]{9a^2}}{\sqrt{3a}}$
23. $\dfrac{\sqrt[5]{5x^4y^3}}{\sqrt[10]{3x^6y^2}}$
24. $\dfrac{\sqrt[4]{2xy^3}}{\sqrt[6]{5x^3y^5}}$

105. Roots of Higher Order (Optional). The three cube roots of the number 8, which are the three roots of the equation $x^3 = 8$, are 2, $(-1 + i\sqrt{3})$, and $(-1 - i\sqrt{3})$. We shall verify these roots by cubing them. Obviously, $2^3 = 8$. Cubing the other two roots,

$$\begin{array}{l}
-1 + i\sqrt{3} \\
\underline{-1 + i\sqrt{3}} \\
1 - i\sqrt{3} \\
\underline{- i\sqrt{3} + 3i^2} \\
1 - 2i\sqrt{3} - 3 \\
\text{or} \\
-2 - 2i\sqrt{3} \\
\underline{-1 + i\sqrt{3}} \\
2 + 2i\sqrt{3} \\
\underline{- 2i\sqrt{3} - 6i^2} \\
2 \qquad\qquad + 6
\end{array}
\qquad\qquad
\begin{array}{l}
-1 - i\sqrt{3} \\
\underline{-1 - i\sqrt{3}} \\
1 + i\sqrt{3} \\
\underline{i\sqrt{3} + 3i^2} \\
1 + 2i\sqrt{3} - 3 \\
\text{or} \\
-2 + 2i\sqrt{3} \\
\underline{-1 - i\sqrt{3}} \\
2 - 2i\sqrt{3} \\
\underline{2i\sqrt{3} - 6i^2} \\
2 \qquad\qquad + 6
\end{array}$$

Thus, $(-1 + i\sqrt{3})^3 = 8$ Thus, $(-1 - i\sqrt{3})^3 = 8$

The cube roots of -8, which are the roots of the equation $x^3 = -8$, are -2, $(1 + i\sqrt{3})$, and $(1 - i\sqrt{3})$. Verify each of these as a root by cubing it.** Any real number (either positive or negative) has three cube roots, one real and two complex. The real root is of the same sign as the number itself and is the principal cube root.

The four roots of the equation $x^4 = 16$, which are the fourth roots of the number 16, are 2, -2, $2i$, and $-2i$. Verify each of these as a root by raising it to the fourth power.** Any positive number has *four* fourth roots, of which two are real (one $+$, one $-$) and two are imaginary numbers. The positive, real, 4th root is the principal 4th root.

The four roots of the equation $x^4 = -16$, which are the 4th roots of -16, are complex numbers. All the 4th roots of any negative number are complex; so that a negative number has no principal 4th root.

In general, any real number (either $+$ or $-$) has *only one* real cube, 5th, 7th, 9th (etc.) root, the others being complex in each case. In general, a *positive* number has two real square, 4th, 6th, 8th (etc.) roots, the others being either imaginary or complex. In general, a *negative* number has only imaginary or complex square, 4th, 6th, 8th (etc.) roots. Since none of its even roots is real, a negative number has no principal even roots.

The cube roots of 1 are 1, $-\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, and $-\frac{1}{2} - \frac{1}{2}i\sqrt{3}$. The cube roots of any positive number can be obtained by multi-

plying each of the cube roots of 1 by the *principal* cube root of the number. For example, since the principal cube root of 8 is 2, its three cube roots are

$$2 \times 1, \qquad 2(-\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3}), \qquad 2(-\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3})$$

or

$$2, \qquad -1 + i\sqrt{3}, \qquad -1 - i\sqrt{3}$$

The principal cube root of 27 is 3; hence the cube roots of 27 are 3, $-\frac{3}{2} + \frac{3}{2}i\sqrt{3}$, and $-\frac{3}{2} - \frac{3}{2}i\sqrt{3}$.

The 4th roots of 1 are 1, -1, i, and $-i$. The principal 4th root of 16 is 2; hence its four 4th roots are 2, -2, $2i$, and $-2i$, found by multiplying each of the 4th roots of 1 by the principal 4th root of 16.

In general, the nth roots of any positive number can be obtained by multiplying the nth roots of 1 by the principal nth root of the number. Likewise, the nth roots of any *negative* number can be obtained by multiplying the nth roots of -1 by the principal nth root of the corresponding positive number. For example, the cube roots of -1 are -1, $\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, and $\frac{1}{2} - \frac{1}{2}i\sqrt{3}$. The cube roots of -8 are obtained by multiplying each of these by $\sqrt[3]{8}$, or 2, giving -2, $1 + i\sqrt{3}$, and $1 - i\sqrt{3}$. A method for determining the approximate values of the principal cube, 4th, and higher roots of positive numbers will be presented in the next chapter.

EXERCISE AND REVIEW QUESTIONS

1. Prove that $-\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ is a cube root of 1.
2. Prove that $-i$ is a 4th root of 1.
3. Prove that i is a 6th root of -1.
4. How many 5th roots has the number 32?
5. What are the 4th roots of 81?
6. Verify your answers to question 5 by raising them to the 4th power.
7. What are the cube roots of 64?
8. What are the cube roots of 125?
9. Verify your answers to the preceding question by cubing them.
10. What are the cube roots of -27?
11. Verify your answers to question 10 by cubing them.
12. What is meant by $x^{\frac{1}{3}}$?
13. What is meant by "the 5th roots of a number N"?
14. If one knows the principal cube root of a number, how can he obtain its other cube roots?

CHAPTER 16

LOGARITHMS

Many of the problems of science and engineering involve arithmetic calculations that are extremely laborious. Such calculations can be simplified a great deal by methods that will be described in this chapter.

106. Calculation by Powers of 10. Consider the problem of evaluating

$$\frac{954 \times 20.7 \times 584}{75.8 \times 482 \times 35.6}$$

To do this by means of arithmetic would be laborious indeed. It can be done very easily, however, if *each number is replaced by the power of* 10 *that is equivalent to it.* The expression then becomes

$$\frac{10^{2.980} \cdot 10^{1.316} \cdot 10^{2.766}}{10^{1.880} \cdot 10^{2.683} \cdot 10^{1.551}}$$

Of course, the question at this point is: "How were $10^{2.980}$, $10^{1.316}$, etc., obtained?" The answer is that the power of 10 equivalent to any given number can be obtained from tables prepared for that purpose, as will be shown later. Once the problem is expressed in terms of powers of 10, the remaining computation is comparatively simple.

The numerator of the expression above is

$$10^{2.980} \cdot 10^{1.316} \cdot 10^{2.766} = 10^{7.062} \text{ (adding the exponents)}$$

and the denominator is

$$10^{1.880} \cdot 10^{2.683} \cdot 10^{1.551} = 10^{6.114} \text{ (adding exponents)}$$

Although $10^{7.062}$ equals approximately 11,530,000 and $10^{6.114}$ equals approximately 1,300,000, their division amounts to a mere subtraction of exponents. Thus,

$$\frac{10^{7.062}}{10^{6.114}} = 10^{(7.062-6.114)} = 10^{0.948}$$

All that remains is to use the same table (from which the powers

of 10 were obtained) to change $10^{0.948}$ to an ordinary number 8.87, which is the result of the calculation. *The entire calculation can be done on a single line:*

$$\frac{954 \times 20.7 \times 584}{75.8 \times 482 \times 35.6} \doteq \frac{10^{2.980} \cdot 10^{1.316} \cdot 10^{2.766}}{10^{1.880} \cdot 10^{2.683} \cdot 10^{1.551}} = \frac{10^{7.062}}{10^{6.114}} = 10^{0.948} \doteq 8.87$$

Note that there are three steps in the procedure:

1. Convert the numbers to powers of 10.

2. Perform the necessary additions and subtractions of exponents.

3. Convert the resulting power of 10 to an ordinary number.

107. Integral Powers of 10. Numbers like 100, 1,000, 10,000, .1, .01, etc., can be expressed as *integral* powers of 10.

Thus

$$\begin{aligned} 10{,}000 &= 10^4 \\ 1{,}000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ .1 &= 10^{-1} \\ .01 &= 10^{-2} \\ .001 &= 10^{-3} \\ .0001 &= 10^{-4} \end{aligned}$$

In the columns at the left, note that numbers *between* 100 and 1000 are equivalent to powers of 10 whose exponents are between 2 and 3, and numbers between 10 and 100 are equivalent to powers of 10 with exponents between 1 and 2, etc. Likewise, *numbers between 1 and 10 are equivalent to powers of 10 whose exponents are between 0 and 1.*

Any positive rational number can be expressed as a number between 1 and 10 multiplied by an integral power of 10:

$$173 = 1.73 \times 10^2, \qquad 2{,}480 = 2.48 \times 10^3,$$
$$.0692 = 6.92 \times 10^{-2}$$

If a carat ($\wedge$) is used to "set off" the number between 1 and 10, the desired integral exponent, called the *characteristic*, is equal to the number of digits between the carat and the decimal point. Thus

$$8_{\wedge}52000. = 8.52 \times 10^5 \qquad 8_{\wedge}52. = 8.52 \times 10^2$$
$$8_{\wedge}5.2 = 8.52 \times 10^1 \qquad 8.52 = 8.52 \times 10^0$$

$.8_{\wedge}52 = 8.52 \times 10^{-1}$ (Note that the characteristic is nega-

tive for numbers smaller than 1.)

$$.08_{\wedge}52 = 8.52 \times 10^{-2} \qquad .0008_{\wedge}52 = 8.52 \times 10^{-4}$$
$$8_{\wedge}5264000 = 8.5264 \times 10^{7} \qquad .0000008_{\wedge}52 = 8.52 \times 10^{-7}$$

Check each of the examples above.

EXERCISES (107)

Write each of the following as a number between 1 and 10 multiplied by an integral power of 10. In each case, determine the characteristic by using the carat.

1. 2,300	2. 64,000	3. 8,439	4. 17.69
5. 8.72	6. 4.65	7. .0634	8. .00726
9. .216	10. .000285	11. 623,400	12. .0000675
13. 62,460,000	14. 175,200	15. .0006142	16. .6284
17. 24,323	18. .0462	19. 31.24	20. .2184

108. Conversion to Powers of 10. The exponents of the powers of 10 equivalent to 1.01, 1.02 . . . 9.98, 9.99, or all the three-digit numbers between 1 and 10, are listed in Table 2, pages 288 and 289. These exponents are irrational numbers whose values can be computed to any desired number of decimal places by methods developed in more advanced courses in mathematics. Their listed values in this book are rounded off to four places.

The use of the table will be illustrated by means of examples:

Example 1. What power of 10 equals 8.72?

Solution: Refer to the table. In the column of numbers at the left, find the first two digits 8.7. Now shift to the column headed by the third digit 2. In this column, even with 8.7, the desired exponent 0.9405 is indicated by 9405. The decimal point and initial zero are omitted in order to save space. Then $8.72 \doteq 10^{0.9405}$.

Example 2. $3.74 \doteq 10^{0.5729}$ (Check this)

Now consider a number outside the range 1 to 10:

Example 3. Convert 6,240 to a power of 10.

Solution: $6_{\wedge}240 = 6.24 \times 10^{3}$

From the table, $6.24 \doteq 10^{0.7952}$

hence $6_{\wedge}240 \doteq 10^{0.7952} \cdot 10^{3} = 10^{3.7952}$

The exponent 3.7952 consists of two parts: the characteristic 3 and the *mantissa* .7952. The mantissa is the exponent of the

power of 10 which corresponds to the number 6.24 set off by the caret, a number between 1 and 10. Since the exponents listed in the table are the mantissas obtained for numbers of all sizes, it is commonly called a *table of mantissas.* Observe the following:

$$6_\wedge 24 = 6.24 \times 10^2 \doteq 10^{0.7952} \cdot 10^2 = 10^{2.7952}$$
$$6_\wedge 2.4 = 6.24 \times 10 \doteq 10^{0.7952} \cdot 10 = 10^{1.7952}$$
$$.006_\wedge 24 = 6.24 \times 10^{-3} \doteq 10^{0.7952} \cdot 10^{-3} = 10^{.7952-3}$$

In the last example, although $10^{.7952-3} = 10^{-2.2048}$, the exponent should be kept in the *binomial form* $.7952 - 3$. This ensures that the decimal part of the exponent will be positive. Always write a negative characteristic *after* the mantissa.*

Example 4. Convert .0327 to a power of 10.

Solution: $.03_\wedge 27 = 3.27 \times 10^{-2}$

From the table, $3.27 \doteq 10^{0.5145}$
Then $.0327 \doteq 10^{.5145-2}$

In converting a number to a power of 10, only the result need be written:

Example 5. $4_\wedge 380 \doteq 10^{3.6415}$

Solution: First, insert the caret and write the characteristic 3. Then refer to the table to find the mantissa corresponding to 4.38, the number set off by the caret. (Check this example.)

Example 6. $.008_\wedge 23 \doteq 10^{.9154-3}$ (Check this)

EXERCISES (108)

Convert to powers of 10:

1. 25.7	2. 746	3. .00382	4. 615,000
5. .128	6. 4870	7. .00111	8. .0215
9. 200	10. .316	11. 63,500,000	12. 100
13. .000618	14. 17,400	15. .0173	16. .899
17. .000073	18. 4,230	19. 2.36	20. .004

* In many books the following convention is used: The exponent $.7952 - 3$ is written as $7.7952 - 10$, while $.2146 - 1$ is written as $9.2146 - 10$. This cumbersome use of a double characteristic, with the negative part always 10, is not recommended by the writer but is mentioned for the sake of completeness. It will be encountered in many standard tables. Actually, of course, $.7952 - 3$ could be written as $1.7952 - 4$, $2.7952 - 5$, or even $11.7952 - 14$.

109. Interpolation. The mantissas listed in the Table 2 are those corresponding to three-digit numbers. In order to determine the mantissas corresponding to numbers with four or more digits, it is necessary to *interpolate.*

Example 1. Convert 2.424 to a power of 10.

Solution: The mantissa corresponding to 2.424 does not appear in the table. The nearest listed values are for $2.42 \doteq 10^{0.3838}$ and

$$2.43 \doteq 10^{0.3856}$$

If $2.424 \doteq 10^x$, then x is between 0.3838 and 0.3856, since 2.424 is between 2.42 and 2.43.

$$\begin{aligned} 2.43 &\doteq 10^{0.3856} \\ 2.424 &= 10^x \\ 2.42 &\doteq 10^{0.3838} \end{aligned}$$

At the left, note that 2.424 is $\frac{4}{10}$ of the way from 2.42 to 2.43. It can be assumed that the exponent x is almost exactly $\frac{4}{10}$ of the way from 0.3838 to 0.3856. The error involved in this assumption is very small.

The process of interpolation is usually carried out without the decimal points. The difference 3856 − 3838, or 18, is called the *tabular difference,* just as in the case of interpolation in the table of square roots. The mantissa corresponding to 2.424 is obtained by adding $\frac{4}{10}$ of the tabular difference to 3838, the smaller of the listed values. Thus $3838 + (.4 \times 18) = 3838 + 7.2 \doteq 3845$, so that $2.424 \doteq 10^{0.3845}$. Note that, since 0.3838 and 0.3856 are values which were rounded off to four places in making up the tables, no mantissa computed from them can be reliably accurate beyond the fourth place. The purpose of interpolation is to determine a mantissa correct to four places (when possible), nothing more. Never retain more decimal places in a mantissa than are given in the table.

When it is possible to do so, interpolate *mentally,* writing only the result. When the number contains no more than four digits, mental interpolation is not difficult.

Example 2. Convert 15,420 to a power of 10.

Solution: (Mentally) The nearest listed mantissas are 1875 and 1903, and the tabular difference is 28. Then $.2 \times 28 = 5.6 \doteq 6$ to the nearest unit. This is added to 1875 to obtain 1881. Then

$$1_\wedge 5420 \doteq 10^{4.1881}$$

A good method to use in developing the ability to interpolate

mentally is this: Always interpolate mentally, even if you merely estimate the result. Then check your mental interpolations with a written calculation. Do this until you become sure of yourself, but *always interpolate mentally* before checking by written calculation. If nothing more, the mental process provides a check that will reveal errors in the written calculation.

Example 3. Convert 7,825,220 to a power of 10.

Solution: The nearest listed mantissas are 8932 and 8938, and the tabular difference is 6. Then $6 \times .522 \doteq 3$. In estimating this product, first use only the fourth digit (5), noting that $6 \times .5 = 3$. Then observe that the fifth digit (multiplied by the tabular difference) is not enough to make the result as large as 3.5; hence the amount to be added is 3. Never go beyond the fifth digit in interpolating.

The desired mantissa is $8932 + 3 = 8935$, so that $7{,}825{,}220 = 10^{6.8935}$.

Example 4. Convert 1,562.43 to a power of 10.

Solution: The listed mantissas for 1.56 and 1.57 are 1931 and 1959, and their difference is 28. Then $28 \times .24 \doteq 7$. Though most students will write out this calculation, it can be done mentally by using

$$.2 \times 28 = 5.6 \text{ and } .04 \times 28 \doteq 1.$$

The total is about $6.6 \doteq 7$, so that

$$1{,}562.43 \doteq 10^{3.1938}.$$

If you always attempt mental interpolation, then check your results, you will soon be able to interpolate mentally in almost all cases. Mental interpolation, beside saving time, serves to develop your powers of visualization.

EXERCISES (109)

Convert to powers of 10:

1. 48.62	2. 874.7	3. 63.741	4. 78.43
5. .006243	6. .72142	7. 124.3	8. 4,862
9. 10,616	10. 6,427	11. .6154	12. 321.7
13. 18.624	14. 864.4	15. 213.2	16. .6182
17. 1,032.7	18. 31.461	19. 98.764	20. .01364

110. Evaluation of Powers of Ten. The result of a calculation by powers of 10 is itself a power of 10, which must be converted to a numerical result. The following examples illustrate the use of the table of mantissas in evaluating powers of 10.

Example 1. The result of a series of calculations is $10^{0.4183}$. Convert this to a numerical answer.

Solution: First, locate the mantissa in the table. In this case, the desired mantissa 4183 is one of those listed, and the corresponding number is 2.62. Since the characteristic is zero, $10^{0.4183} \doteq 2.62$.

Example 2. Evaluate $10^{3.8965}$.

Solution: The mantissa 8965 corresponds to the number 7.88; therefore $10^{0.8965} = 7.88$, or $10^{3.8965} = 788 \times 10^3 = 7_\wedge 880$. If the number 7.88 is recorded as $7_\wedge 88$, using the carat, only the result need be written. Thus, one writes only $10^{3.8965} \doteq 7_\wedge 880$, the characteristic (3) being used to locate the decimal point three places to the right of the caret.

Example 3.	$10^{.6274-2} \doteq .04_\wedge 24$	(check this)
Example 4.	$10^{.4116-3} \doteq .002_\wedge 58$	(check)

When the mantissa is not one of those listed in the table, interpolation is necessary.

Example 5. Evaluate $10^{1.7585}$.

Solution: The nearest listed mantissas are 7582 and 7589, corresponding to 5.73 and 5.74, respectively. The first three digits of the desired number are those corresponding to the smaller listed mantissa, and the fourth digit is obtained by interpolation. The tabular difference is $7589 - 7582 = 7$, and 7585 exceeds 7582 by 3; hence it is three-sevenths of the way from 7582 to 7589. The fourth digit of the desired number is obtained (mentally) from $\frac{3}{7} \doteq .43 \doteq .4$. Then

$$10^{1.7585} \doteq 5_\wedge 7.34$$

Example 6. Evaluate $10^{.5519-3}$.

Solution: The nearest listed mantissas are 5514 and 5527, corresponding to 3.56 and 3.57, respectively. The tabular difference is 13, and 5519 exceeds 5514 by 5; hence the fourth digit is obtained from $\frac{5}{13} \doteq .4$. Then $10^{.5519-3} \doteq .003_\wedge 564$.

The fourth digit should be determined mentally in almost all cases. Though it may seem difficult to do this at first, the technique can be acquired rapidly. A four-place mantissa cannot be used to obtain five digits of the corresponding number; hence *do not record more than four digits* (other than zero).

EXERCISES (110)

Evaluate to four figures:

1. $10^{2.9552}$	2. $10^{3.8162}$	3. $10^{.5623-1}$	4. $10^{.2201-4}$
5. $10^{4.6085}$	6. $10^{1.3655}$	7. $10^{.8041-2}$	8. $10^{0.9263}$

9. $10^{1.9214}$	10. $10^{2.9436}$	11. $10^{.7508-1}$	12. $10^{0.8587}$
13. $10^{2.5709}$	14. $10^{0.4567}$	15. $10^{.2930-3}$	16. $10^{1.3481}$
17. $10^{.7150-2}$	18. $10^{4.5000}$	19. $10^{.5481-4}$	20. $10^{.8708-2}$

111. Application of the Method. Calculation by powers of 10 involves three steps, as follows:

1. Convert the numbers to powers of 10.
2. Perform the necessary operations, on the exponents.
3. Convert the resulting power of 10 to an ordinary number.

Example 1. Evaluate $\dfrac{746 \times 632}{561 \times 34.3}$.

Solution:

$$\frac{7_\wedge 46 \times 6_\wedge 32}{5_\wedge 61 \times 3_\wedge 4.3} \doteq \frac{10^{2.8727} \cdot 10^{2.8007}}{10^{2.7490} \cdot 10^{1.5353}} = \frac{10^{5.6734}}{10^{4.2843}} = 10^{1.3891} \doteq 2_\wedge 4.49$$

Example 2. Evaluate $\dfrac{81.3 \times 56.7}{516 \times 28.3}$.

Solution:

$$\frac{8_\wedge 1.3 \times 5_\wedge 6.7}{5_\wedge 16 \times 2_\wedge 8.3} \doteq \frac{10^{1.9101} \cdot 10^{1.7536}}{10^{2.7126} \cdot 10^{1.4518}} = \frac{10^{3.6637}}{10^{4.1644}} = 10^{.4993-1} \doteq .3_\wedge 157$$

Note that the negative characteristic obtained in subtracting exponents is shifted to its proper position, which is *after* the mantissa:

$$\begin{array}{r} 3.6637 \\ 4.1644 \\ \hline .4993 - 1 \end{array}$$

Example 3. Evaluate $\sqrt[5]{8{,}120}$.

Solution: $\sqrt[5]{8_\wedge 120} \doteq \sqrt[5]{10^{3.9096}} = 10^{\frac{3.9096}{5}} \doteq 10^{0.7819} \doteq 6.051$

If the characteristic is negative, it must be made evenly divisible when extracting a root:

Example 4. Evaluate $\sqrt[4]{.0000562}$.

Solution: $\sqrt[4]{.00005_\wedge 62} \doteq \sqrt[4]{10^{.7497-5}} = 10^{\frac{.7497-5}{4}}$

Since the negative characteristic -5 is not evenly divisible by 4, it is best to subtract 3 (in this case) from the characteristic (making it -8)

and add 3 to the mantissa. Thus,

$$\sqrt[4]{.00005_{\wedge}62} \doteq 10^{\frac{.7497-5}{4}} = 10^{\frac{3.7497-8}{4}} \doteq 10^{.9376-2} \doteq .08_{\wedge}662$$

In general, avoid negative fractions in the exponent by subtracting enough from the negative characteristic to make it *evenly divisible* and adding the same amount to the mantissa.

Example 5. Evaluate $\sqrt[3]{.0487}$.

Solution: $\sqrt[3]{.04_{\wedge}87} \doteq 10^{\frac{.6875-2}{3}} = 10^{\frac{1.6875-3}{3}} = 10^{.5625-1} \doteq .3_{\wedge}652$

Example 6. Evaluate $\dfrac{74.2\sqrt[3]{.561}}{\sqrt{.817}}$.

Solution: $\dfrac{7_{\wedge}4.2\sqrt[3]{.5_{\wedge}61}}{\sqrt{.8_{\wedge}17}} \doteq \dfrac{10^{1.8704}\cdot 10^{\frac{2.7490-3}{3}}}{10^{\frac{1.9122-2}{2}}} \doteq \dfrac{10^{1.8704}\cdot 10^{.9163-1}}{10^{.9561-1}}$

$$= \frac{10^{1.7867}}{10^{.9561-1}} = 10^{1.8306} \doteq 6_{\wedge}7.70$$

Check this problem in every detail. In doing so, you will encounter several situations that should be examined carefully.

EXERCISES (111)

Evaluate to four figures:

1. 25.7×31.8
2. 74.7×6.73
3. $56.5 \times 931 \times .0625$
4. 32.8×67.34
5. 65.2×11.754
6. $156.27 \times .452$
7. $\frac{852}{278}$
8. $\dfrac{637.4}{.285}$
9. $\dfrac{.00564}{.0742}$
10. $\dfrac{.0127423}{21.4}$
11. $\dfrac{.6172}{2480}$
12. $\dfrac{81.5 \times 487}{936}$
13. $\dfrac{.0784 \times 27.9}{846.2}$
14. $\dfrac{3.20 \times 524}{.003422}$
15. $\dfrac{436 \times 3{,}275}{626 \times 148.74}$
16. $\dfrac{291 \times 82.9}{562 \times 131.43}$
17. $\dfrac{91.2 \times 68.8 \times 381}{223 \times 57.28}$
18. $\dfrac{256 \times 431 \times 84.7}{811 \times .378 \times 53.92}$
19. $\dfrac{954 \times 20.7 \times 584}{75.8 \times 482 \times 35.6}$
20. $\dfrac{6.573^2 \times 17.5}{2.47}$
21. $\dfrac{2.65^3 \times 2.325}{4.27^2 \times 31.42}$
22. $\dfrac{1.324^3 \times 2.621^2}{44 \times 127.25}$
23. $387 \times \sqrt[3]{24{,}820}$
24. $\sqrt[5]{63{,}742.2}$
25. $287\sqrt[4]{48.28}$

26. $\dfrac{284}{\sqrt{1,570}}$ 27. $\dfrac{48.74}{\sqrt[5]{24,600}}$ 28. $\dfrac{\sqrt{642}}{\sqrt{551}}$

29. $\dfrac{25.7\sqrt{8,270}}{\sqrt[3]{85.2}}$ 30. $\dfrac{\sqrt{42.5}\cdot\sqrt[3]{657}}{275.3}$ 31. $\sqrt{\dfrac{6,570}{.0372 \times .451}}$

32. $\dfrac{\sqrt[3]{.04257}}{44.2\sqrt{.162}}$ 33. $\dfrac{272\sqrt{.00562}}{\sqrt{.427}}$ 34. $\dfrac{2,955\sqrt{462}}{331\sqrt{2,172}}$

35. $\dfrac{(28.7)^2\sqrt{175}}{\sqrt{4,827}}$ 36. $\dfrac{\sqrt{34}\cdot\sqrt[3]{742.4}}{\sqrt{.081}\cdot\sqrt{41.5}}$

112. Logarithms. When the method of computation by powers of 10 is thoroughly familiar, one should abbreviate the procedure by writing only the exponents involved and, of course, the result of the computation.

Example 1. Evaluate $\dfrac{817 \times 462}{937}$.

Complete Computation

$$\frac{817 \times 462}{937.4} \doteq \frac{10^{2.9122}\cdot 10^{2.6646}}{10^{2.9719}}$$
$$= \frac{10^{5.5768}}{10^{2.9719}} = 10^{2.6049} \doteq 4_\wedge 02.6$$

Abbreviated Notation

$$\left.\begin{array}{rr} & 2.9122 \\ & 2.6646 \\ \text{(Sum)} & 5.5768 \\ & 2.9719 \\ \text{(Difference)} & 2.6049 \end{array}\right\}\text{exponents}$$

Ans.: $4_\wedge 02.6$ (number)

Notice that in the abbreviated notation the exponents are in positions convenient for adding and subtracting. This makes for increased speed and accuracy.

In extended computations, the use of the abbreviated notation may lead to some confusion as to which exponents correspond to which numbers or as to which exponents should be added, subtracted, etc. For this reason, the exponent of the power of 10 equivalent to a given number is called the *logarithm* of that number.* The word *logarithm* serves as an "identification tag" that carries the number corresponding to the given exponent. In logarithmic notation, the example above is written as follows:

* Specifically, it is called the logarithm *to the base* 10 (or the *common* logarithm) of the number, since bases other than 10 are sometimes used.

$\log 8_\wedge 17 \doteq 2.9122$	In the computation at the left, writing $\log 8_\wedge 17 \doteq 2.9122$ serves two important purposes:
$\log 4_\wedge 62 \doteq 2.6646$	1. It identifies 2.9122 as an exponent rather than an ordinary number.
$\log \text{numerator} \doteq 5.5768$	2. It associates 2.9122 with the number 817 from which it was obtained.
$\log 9_\wedge 37.4 \doteq 2.9719$	
$\log \text{of answer} \doteq 2.6049$	
Ans. $\doteq 4_\wedge 02.6$	

Compare this solution, in detail, with the complete computation written out in powers of 10.

The *logarithmic* notation just described makes it easy to "keep straight" in computation without writing out the complete powers of 10. At the same time, it permits arrangement of the exponents in vertical columns convenient for addition and subtraction. From now on the logarithmic notation will be employed, and the exponents used in the calculation will be called logarithms rather than exponents. It must be remembered, however, that the logarithm of a number is nothing but the exponent used in expressing the number as a power of 10.

Example 2. Evaluate log 7,265.

Solution: $7_\wedge 265 \doteq 10^{3.8612}$; hence $\log 7.265 \doteq 3.8612$

Example 3. Evaluate log .06284.

Solution: $.06_\wedge 284 = 10^{.7983-2}$; hence $\log .06284 \doteq .7983 - 2$

In evaluating the logarithm of a number, determine the characteristic and mantissa by the same methods used in converting to powers of 10. It is not necessary to write or think of the power of 10 itself.

Example 4. $\log 175 \doteq 2.2430$ (check this)
Example 5. $\log .2675 \doteq .4273 - 1$ (check)

Note that the exponents listed in the table of mantissas are the logarithms of the numbers between 1 and 10, or the mantissas of the logarithms of numbers outside the range 1 to 10.

At the end of a series of logarithmic calculations, the result is obtained in the form of a logarithm (*i.e.*, the exponent of a power of 10). It is then necessary to evaluate the corresponding *number*, which is the desired answer. This answer, or number, is

called the *antilogarithm*. To find the number corresponding to a given logarithm, evaluate 10^x, where x is the logarithm:

Example 6. Evaluate antilog 2.8931.

Solution: antilog $2.8931 = 10^{2.8931} \doteq 7_\wedge 81.8$

Note that "antilog 2.8931 = 781.8" means, "The number whose logarithm is 2.8931 equals 781.8." This notation identifies 781.8 as a *number* rather than a logarithm and, at the same time, links 781.8 with the *logarithm from which it was obtained.*

Example 7. Evaluate antilog $.6404 - 3$.

Solution: antilog $.6404 - 3 = 10^{.6404-3} \doteq .004_\wedge 369$

There is no necessity for writing the power of 10. Simply determine the mantissa of the logarithm and write the corresponding number, using the carat and characteristic to locate the decimal point.

Example 8. antilog $4.2878 \doteq 1_\wedge 9{,}400$ (check)
Example 9. antilog $.6728 - 2 \doteq .04_\wedge 708$ (check)

EXERCISES (112)

Evaluate:

1. log 222	2. log 1760	3. log .00747
4. log .0562	5. log 8.174	6. log .06613
7. log 63,527,000	8. log .6173	9. log .4621
10. log 2.752	11. log 5.727	12. log .4216
13. log 65.27	14. log .06272	15. log .01724
16. log 414.63	17. antilog 3.4217	18. antilog 2.8995
19. antilog .5684 − 2	20. antilog 5.8605	21. antilog .2639 − 3
22. antilog .6738 − 2	23. antilog .5289 − 1	24. antilog .4349 − 5
25. antilog 0.8487	26. antilog 1.9574	27. antilog 2.0079
28. antilog .0162 − 1	29. antilog 2.7382	30. antilog .0127 − 4

113. Calculation by Logarithms. Calculation by logarithms consists of three steps:

1. Determine the logarithms of the numbers involved.
2. Combine the logarithms to obtain the logarithm of the answer.
3. Evaluate the corresponding antilogarithm (number).

Example 1. Evaluate $49.3 \times \sqrt{66.4}$.

Solution:

$$\begin{aligned} \log 4_\wedge 9.3 &\doteq 1.6928 \\ \log \sqrt{66.4} = \tfrac{1}{2}\log 6_\wedge 6.4 \doteq \tfrac{1}{2}(1.8222) &= \underline{0.9111} \\ \log N &\doteq 2.6039 \end{aligned}$$

where N is the desired product. Then

$$N \doteq \text{antilog } 2.6039 \doteq 4_\wedge 01.7$$

Write out the corresponding solution in powers of 10, comparing the two notations in detail.**

In all except the simplest calculations, it is important to plan the procedure in detail. The plan can be set up most conveniently by means of the laws of logarithms, which are the laws of exponents translated into the logarithmic notation.

Law I: $$\log (A \cdot B) = \log A + \log B$$

Proof: Suppose $A = 10^a$ and $B = 10^b$ (in this and the following examples). Then $a = \log A$ and $b = \log B$. But

$$A \cdot B = 10^a \cdot 10^b = 10^{a+b}$$

proving that $\log (A \cdot B) = \log A + \log B$. This law can be extended to include any number of factors; *viz.*,

$$\log (A \cdot B \cdot C) = \log A + \log B + \log C$$

Example 2. Evaluate $5.61 \times 93.7 \times 7.36$.

Solution: $\log (5.61 \times 93.7 \times 7.36) = \log 5.61 + \log 93.7 + \log 7.36$

$$\begin{aligned}
\log 5.61 &\doteq 0.7490 \\
\log 9_\wedge 3.7 &\doteq 1.9717 \\
\log 7.36 &\doteq 0.8669 \\
\hline
\log N &\doteq 3.5876 \\
N &\doteq \text{antilog } 3.5876 \doteq 3_\wedge 869
\end{aligned}$$

Note that the first step is to express the logarithm of the quantity to be evaluated, in terms of the logarithms of the individual numbers.

Law II: $$\log \frac{A}{B} = \log A - \log B$$

Proof: $\dfrac{A}{B} = \dfrac{10^a}{10^b} = 10^{a-b};$ hence $\log \dfrac{A}{B} = a - b$

or $$\log \frac{A}{B} = \log A - \log B$$

Example 3. Evaluate $\dfrac{857}{33.5}$.

Solution: $$\log \frac{857}{33.5} = \log 857 - \log 33.5$$

$$\begin{aligned} \log 857 &\doteq 2.9330 \\ \underline{\log 33.5} &\doteq \underline{1.5250} \\ \log N &\doteq 1.4080 \\ N &\doteq \text{antilog } 1.4080 \doteq 2_\wedge 5.59 \end{aligned}$$

Law III: $$\log A^n = n \log A$$

Proof: $A^n = (10^a)^n = 10^{na}$; hence $\log A^n = na$,

or $$\log A^n = n \log A$$

Example 4. Evaluate 7.37×2.91^3.

Solution: $$\log (7.37 \times 2.91^3) = \log 7.37 + 3 \log 2.91$$

$$\begin{aligned} \log 7.37 &\doteq 0.8675 \\ 3 \log 2.91 = 3(0.4639) &\doteq 1.3917 \\ \log N &\doteq 2.2592 \\ N &\doteq \text{antilog } 2.2592 \doteq 1_\wedge 81.6 \end{aligned}$$

Law IV: $$\log \sqrt[n]{A} = \frac{1}{n} \log A$$

Proof: $\sqrt[n]{A} = \sqrt[n]{10^a} = 10^{\frac{a}{n}}$; hence $\log \sqrt[n]{A} = \frac{a}{n}$,

then $$\log \sqrt[n]{A} = \frac{1}{n} \log A$$

Example 5. Evaluate $87.2 \times \sqrt[4]{487}$.

Solution: $$\log (87.2 \times \sqrt[4]{487}) = \log 87.2 + \tfrac{1}{4} \log 487$$

$$\begin{aligned} \log 8_\wedge 7.2 &\doteq 1.9405 \\ \tfrac{1}{4} \log 4_\wedge 87 \doteq \tfrac{1}{4}(2.6875) &\doteq 0.6719 \\ \log N &\doteq 2.6124 \\ N &\doteq \text{antilog } 2.6124 \doteq 4_\wedge 09.7 \end{aligned}$$

The following examples illustrate the use of the laws of logarithms and the way in which the plan of calculation should be set up. In each case the first step is to express the logarithm of the quantity to be evaluated in terms of the logarithms of its parts. In brief, calculation by logarithms consists simply of finding the logarithm of the answer (from the logarithms of the numbers involved) and then determining the corresponding number, or antilogarithm.

Example 6. Evaluate $\sqrt{332} \times .174^2 \times \sqrt[3]{36.7}$.

Solution: $\log N = \frac{1}{2} \log 332 + 2 \log .174 + \frac{1}{3} \log 36.7$

where N is the quantity to be determined.

$$\begin{array}{rl}
\frac{1}{2}\log 332 \doteq \frac{1}{2}(\quad) = & (\quad) \\
2\log .174 \doteq 2(\quad) = & (\quad) \\
\frac{1}{3}\log 36.7 \doteq \frac{1}{3}(\quad) = & (\quad) \\
\hline
\log N \doteq & (\quad) \\
N \doteq \text{antilog}\ (\quad) \doteq & 183.3
\end{array}$$

Complete the solution.**

Example 7. Evaluate $\dfrac{832 \times 453}{622 \times 384}$.

Solution: $\log N = \log \text{numerator} - \log \text{denominator}$
$= (\log 832 + \log 453) - (\log 622 + \log 384)$

$$\begin{array}{rl}
\log 832 \doteq & (\quad) \\
\log 453 \doteq & (\quad) \\
\hline
\log \text{numerator} \doteq & (\quad) \\
\log \text{denominator} \doteq & (\quad) \\
\hline
\log N \doteq & (\quad) \\
N \doteq \text{antilog}\ (\quad) \doteq & 1.577
\end{array}
\qquad
\begin{array}{rl}
\log 622 \doteq & (\quad) \\
\log 384 \doteq & (\quad) \\
\hline
\log \text{denominator} \doteq & (\quad)
\end{array}$$

Complete the calculation.**

Example 8. Evaluate $\dfrac{33.7\sqrt[3]{.0823}}{12.4\sqrt{743}}$.

Solution: $\log N = \log \text{numerator} - \log \text{denominator}$
$= (\log 33.7 + \frac{1}{3}\log .0823) - (\log 12.4 + \frac{1}{2}\log 743)$

$$\begin{array}{rl}
\log 33.7 \doteq & 1.5276 \\
\frac{1}{3}\log (.08_\wedge 23) \doteq \frac{1}{3}(.9154 - 2) = \frac{1}{3}(1.9154 - 3) \doteq & .6385 - 1 \\
\hline
\log \text{numerator} \doteq & 1.1661
\end{array}$$

$$\begin{array}{rl}
\log 1_\wedge 2.4 \doteq & 1.0934 \\
\frac{1}{2}\log 7_\wedge 43 \doteq \frac{1}{2}(2.8710) = & 1.4355 \\
\hline
\log \text{denominator} \doteq & 2.5289
\end{array}$$

$$\begin{array}{rl}
\log \text{numerator} \doteq & 1.1661 \\
\log \text{denominator} \doteq & 2.5289 \\
\hline
\log N \doteq & .6372 - 2 \\
N \doteq \text{antilog}\ (.6372 - 2) \doteq & .04_\wedge 337
\end{array}$$

Check this example in detail.**

When a calculation involves one or more negative numbers, determine the sign of the result by inspection, then perform the computation with all numbers positive. Thus, to evaluate

27 (−14.2), evaluate 27 × 14.2 by logarithms, and prefix a negative sign to the result. *A negative number has no logarithm.*

EXERCISES (113)

Evaluate by logarithms. Plan each solution completely, leaving spaces for the logarithms:

1. 43.5×1.67
2. 38.9×7.24
3. $53.6 \times .627$
4. 42.17×8.7
5. 8.434×10.4
6. 239.4×3.84
7. $1.72 \times 44.62 \times 21.7$
8. $.017 \times 44.6 \times 28.2$
9. $12.45 \times 6.32 \times 14.47$
10. $\dfrac{36.8}{2.16}$
11. $\dfrac{2{,}442}{41.8}$
12. $\dfrac{2.65}{414.5}$
13. $\dfrac{.000172}{44.23}$
14. $\dfrac{.01272}{365}$
15. $\dfrac{64{,}400}{29{,}140}$
16. $\dfrac{4.16 \times 31.7}{482}$
17. $\dfrac{3.27 \times .058}{362}$
18. $\dfrac{44.7 \times 32.35}{47.1}$
19. $\dfrac{84.6 \times .142}{.0165 \times 21.72}$
20. $\dfrac{527 \times .0592}{7.2 \times 3.448}$
21. $\dfrac{61.35 \times 4.263}{91.82 \times 25.6}$
22. $\dfrac{68.4 \times 71.2 \times .17}{38.4 \times 425 \times 2.32}$
23. $\dfrac{31.82 \times 7.6 \times 41.9}{3.43 \times 7.82 \times 68}$
24. $\dfrac{954 \times 20.7 \times 584}{75.8 \times 482 \times 35.6}$
25. 17.8×4.42^3
26. $24.8^3 \times .0176^2$
27. 2.425^6
28. $\sqrt[3]{45{,}700}$
29. $\sqrt{44.6}$
30. $\sqrt[3]{.0276}$
31. $\sqrt[5]{42{,}700}$
32. $2.75\sqrt[4]{.05952}$
33. $44.7\sqrt{1780}$
34. $\dfrac{\sqrt{647}}{\sqrt{551.7}}$
35. $\dfrac{284}{\sqrt{1670}}$
36. $\dfrac{\sqrt{42.7}\sqrt[3]{.597}}{432}$
37. $\dfrac{25.8\sqrt{8{,}280}}{\sqrt[3]{85.3}}$
38. $\dfrac{\sqrt[3]{.042372}}{44.2\sqrt{.162}}$
39. $\sqrt{\dfrac{6{,}570}{.372 \times .0451}}$

114. Accuracy. In most cases the result of a computation done by means of four-place tables of mantissas will be correct to four figures. Quite often, however, the fourth figure is in error by one unit. For example, using four-place tables, one would obtain $2 \times 3 \doteq 5.999$ (check this). Once in a great while the fourth figure will be in error by 2 units. In order to obtain results even this accurate, it is necessary to use the fifth figure of a number (if it has one) in interpolating, if the tabular difference is greater than 20. The theoretically possible error is much greater but exists only as a remote possibility.

More accurate results can be obtained by using tables containing mantissas that are accurate to five or more places. It should be understood that tables with five-place mantissas would contain the mantissas corresponding to all the four-digit numbers between 1 and 10, *i.e.*, 1.001, 1.002, etc.; otherwise the tabular differences would be so large as to make interpolation very laborious.

115. Logarithms to Any Base. In the relation $y = a^x$, the exponent x is defined as the logarithm of y to the base a. The equation $\log_a y = x$, which is read, "the logarithm of y to the base a equals x," expresses exactly the same relation as does $y = a^x$. The equation $\log_a y = x$ is called the *logarithmic* form of the relation, as distinguished from the *exponential* form $y = a^x$. Several such relations are given in the following table, in both exponential and logarithmic forms.

Exponential form	$2^3 = 8$	$9^{1.5} = 27$	$3^4 = 81$	$10^2 = 100$	$7^2 = 49$
	or	or	or	or	or
Logarithmic form	$\log_2 8 = 3$	$\log_9 27 = 1.5$	$\log_3 81 = 4$	$\log 100 = 2$	$\log_7 49 = 2$

By the *base* of the logarithm is meant the base with which the exponent is used in the exponential notation. When the base is not written, the base 10 is understood.

The only kinds of logarithms widely used in computation are common logarithms (base 10) and so-called *natural* logarithms (base $e = 2.7182+$). Natural logarithms are more convenient for certain special applications in science and engineering but are not used nearly so much as common logarithms.

The significance of any logarithmic relation is best seen by writing it in its exponential form. Thus $\log_2 16 = x$ becomes $2^x = 16$; whence it can be seen that $x = 4$, since $2^4 = 16$.

EXERCISES (115)

Write in exponential notation, and evaluate the unknown:

1. $\log_4 16 = x$
2. $\log_2 32 = x$
3. $\log 1{,}000 = x$
4. $\log_5 125 = x$
5. $\log_3 y = 2$
6. $\log y = 4$
7. $\log_3 y = 4$
8. $\log_2 y = 6$
9. $\log_2 128 = x$
10. $\log_a 9 = 2$
11. $\log_a 343 = 3$
12. $\log_a 121 = 2$
13. $\log_{100} y = \frac{1}{2}$
14. $\log_{625} y = \frac{1}{4}$
15. $\log_a 1{,}000 = \frac{3}{2}$

Write in logarithmic notation:

16. $3^4 = 81$ 17. $a^b = c$ 18. $2^{m+1} = n$

19. $5^4 = 625$ 20. $7^x = 482$ 21. $4^{x+1} = y - 7$

22. $m^{2-y} = n$ 23. $(r - s)^t = z$

24. $(m - n)^{p-q} = x + y$

116. Exponential Equations (Optional). An *exponential equation* is one in which the unknown occurs as (or in) an exponent. Many such equations can be solved at sight.

Example 1. $2^x = 32$

Solution: Since $32 = 2^5$, it is seen that x is 5. This method can be used only when the particular power of 2 equivalent to the number (32 in this case) can be determined by inspection. A more systematic procedure is to evaluate the logarithm (to the base 10) of each side of the equation; *viz.*,

$$\log (2^x) = \log 32$$
$$x \log 2 = \log 32$$
$$x = \frac{\log 32}{\log 2} \doteq \frac{1.5051}{0.3010} \doteq 5$$

Note that log 32 is *divided by* log 2; hence do not make the common error of subtracting.

Example 2. $3^{x+1} = 7$

$$(x + 1) \log 3 = \log 7$$
$$x + 1 = \frac{\log 7}{\log 3} \doteq \frac{0.8451}{0.4771}$$

The quotient $\frac{.8451}{.4771}$ can be evaluated by long division as 1.79. Then $x + 1 \doteq 1.79$, or $x \doteq 1.79 - 1 = .79$.

It is preferable to use logarithms to divide .8451 by .4771; *viz.*,

$$\begin{array}{rl} \log .8451 \doteq & .9315 - 1 \\ \log .4771 \doteq & .6785 - 1 \\ \hline \log \text{quotient} \doteq & 0.2530 \\ \text{quotient} \doteq & 1.79 \end{array}$$

Then $x + 1 \doteq 1.79$, or $x \doteq 1.79 - 1 = .79$

In order to avoid confusion, when you have reached the equation $x + 1 \doteq \frac{0.8451}{0.4771}$, *forget* how 0.8451 and 0.4771 were obtained, and divide them as numbers by evaluating their logarithms and subtracting, etc.

Care must be taken when negative logarithms are involved:

Example 3. $$867^x = .035$$

$$x \log 867 = \log .035$$

$$x = \frac{\log .035}{\log 867} \doteq \frac{.5441 - 2}{2.9380}$$

To evaluate this ratio, we must change .5441 − 2 to a single number (which will be negative).

Then $$x \doteq \frac{-1.4559}{2.9380}$$ (note minus sign)

$$\begin{aligned} \log 1.4559 &\doteq 0.1632 \\ \log 2.9380 &\doteq 0.4680 \\ \hline \log \text{quotient} &= .6952 - 1 \\ \text{quotient} &= .4_{\wedge}957 \end{aligned}$$

hence $$x = -.4957$$

EXERCISES (116)

Solve for x:

1. $5^x = 272$
2. $2^x = 1{,}850$
3. $6^x = 5{,}270$
4. $15^x = 20$
5. $7^x = 250$
6. $25^x = 6{,}970$
7. $3^{2x} = 75$
8. $2.72^x = 14.4$
9. $23.2^{3x} = 64{,}500$
10. $5^{x+1} = 372$
11. $2.85^{x+1} = 84.7$
12. $6.27^{x-1} = 447$
13. $7.5^{2x-3} = 957$
14. $4.42^{x+3} = 8{,}750$
15. $4.75^x = .0265$
16. $3.84^{2x-1} = .0295$
17. $15.5^{x-2} = .126$
18. $78.7^{x-1} = .285$
19. $3.5^{x-1} = 2.7^{x+1}$
20. $72^x = 5.5^{x+3}$
21. $150^{x-1} = 7^{2x+3}$
22. $27.5^{x-1} = 7.7^{2x+1}$
23. $9.72^x = 5.4^{-3}$
24. $27.2^{x-1} = 6.51^{-5}$

REVIEW QUESTIONS

1. Describe the process of evaluating the product $A \cdot B \cdot C$ by means of powers of 10.
2. Describe the process of evaluating the product $A \cdot B \cdot C$ by means of logarithms.
3. What is meant by $\log_a y$? by $\log y$?
4. What is an exponential equation?
5. Describe how an exponential equation is solved.

CHAPTER 17

RATIO, PROPORTION, AND VARIATION

The terms *ratio*, *proportion*, and *variation* are used to express *in words* some of the relationships already encountered by the student in the form of simple equations. These terms, and the relationships that they identify, are so universal in application that an understanding of their use is needed in practically all fields of endeavor.

117. Ratio. One of the simplest ways of comparing two real numbers is to state their *ratio*, which is their indicated quotient.

Example 1. The ratio of 5 to 8 is $\frac{5}{8}$, or 5:8 (read "5 to 8.")

It is customary to reduce a ratio to its lowest terms:

Example 2. The ratio of 12 to 10 is $\frac{12}{10}$, or $\frac{6}{5}$

The numbers associated with *like quantities* can be compared by ratio. In this sense only, the quantities themselves may be compared by ratio.

Example 3. The ratio of 7 ft. to 9 ft. is $\frac{7}{9}$

Since a ratio involves only numbers, two quantities must have the same units in order to be compared by ratio.

Example 4. The ratio of 18 in. to 4 ft. is $\frac{18 \text{ in.}}{48 \text{ in.}}$, or $\frac{3}{8}$. It is convenient to think of the units (inches in this case) as canceling to produce a numerical ratio. Note that it is necessary to express the quantities in the same units before their ratio can be evaluated.

Example 5. The ratio of 15 oz. to $3\frac{1}{2}$ lb. is $\frac{15}{3\frac{1}{2} \times 16} = \frac{15}{56}$.

Confusion is often encountered in comparing *unlike* quantities by ratio.

Example 6. An alloy contains 45 oz. of zinc, and 80 oz. of copper. What is the ratio of zinc to copper?

It must be remembered that a ratio is a comparison of *numbers*. It cannot be used to compare the metals zinc and copper, but it can be used to compare the *number* of ounces of zinc to the *number* of ounces of copper, or the number of cubic inches of zinc to the number of cubic inches of copper. For this reason, when *unlike* quantities are compared by ratio, the basis of comparison must be stated. It is meaningless to ask, "What is the ratio of zinc to copper in this alloy?"; but it is proper to say, "The ratio of zinc to copper, *by weight*, is $\frac{45 \text{ oz.}}{80 \text{ oz.}}$, or $\frac{9}{16}$," since a unit of weight is used as the basis of numerical comparison in this case.

Example 7. An audience includes 250 men, 150 women, and 75 children. What is the ratio of men to women? of children to women?

Remembering that a ratio is a comparison of numbers, we see that the "ratio of men to women" does not compare men to women, but the *number* of men to the *number* of women. Whenever the basis of comparison between unlike quantities is not stated, it is assumed to be a comparison by *number* (of units). In this example, the ratio (by number) of men to women is $\frac{250}{150}$, or 5.3. The ratio (by number) of children to women is $\frac{75}{150}$, or 1:2. Note that the units will not cancel; therefore it is meaningless to write $\frac{75 \text{ children}}{150 \text{ women}} = ?$ One can compare only the numbers involved in unlike quantities. Such a comparison is possible only if the unlike quantities to be compared are made up of discrete units, or can be thought of in terms of units.

The ratio of a part to the whole may be expressed as a ratio, but if the part is not like the whole, the basis of comparison must be stated. The ratio of a part to the whole may be expressed in per cent.

Example 8. A mixture contains 3 qt. of alcohol and 5 qt. of water. What part of the mixture is alcohol?

Solution: The mixture consists of 8 qt. in all; hence the ratio of the *volume* of alcohol to the *volume* of the entire mixture is $\frac{3}{8}$, or 37.5%. The alcoholic content of the mixture is said to be 37.5% *by volume.*

Suppose that we wish to use weight (instead of volume) as the basis of comparison in the preceding example. If alcohol weighs 1.67 lb. per quart and water 2.08 lb. per quart, the total weight is

$(3 \times 1.67) + (5 \times 2.08) = 5.01 + 10.40 = 15.41$ lb. The ratio of the weight of alcohol to that of the entire mixture is $\frac{5.01}{15.41}$, or .325; hence the alcoholic content is said to be 32.5% by weight. Note that it is absolutely necessary to state the basis of comparison; for the alcoholic content in this case is 37.5% by volume but only 32.5% by weight.

A *continued* ratio is written in the form $a:b:c$. Thus, 5 ft., 7 ft., and 11 ft. are in the ratio 5:7:11, which is read, "5 to 7 to 11"; while 6 lb., 9 lb., and 15 lb. are in the ratio 6:9:15, or 2:3:5.

EXERCISES AND PROBLEMS (117)

Express each of the following ratios in its simplest form:

1. 36 to 42
2. 56 to 60
3. 42:24
4. 144:60
5. 45 in.:18 in.
6. $2\frac{1}{2}$ ft.:30 ft.
7. 7 lb.:42 oz.
8. 200 yd.:1 mile:2 miles
9. 14 in.:2 ft.:3 ft.
10. 7 qt.:3 gal.:2 gal.
11. 250 cm.:10 ft. (Use 1 in. 2.54 cm.)
12. 4,840 sq. yd.:5 acres
13. 10 kg.:60 lb. (Use 1 kg. = 2.2 lb.)

14. Find the ratio, by area, of a rectangle 4 by 6 in. to a rectangle 3 by 10 in.

15. Find the ratio, by area, of a 6-in. square to an 8-in. square.

16. In making a certain type of steel, 1 ton of iron is used with 25 lb. of carbon and 80 lb. of manganese. Express the carbon content and the manganese content in per cent (by weight).

17. A feed mixture contains 3 bu. of shelled corn, 5 bu. of barley, and 7 bu. of oats. Express the shelled-corn content and the barley content in per cent (by volume) of the entire mixture.

18. In Prob. 17, what is the ratio of corn to barley (by volume)? of barley to oats? Express the three in a continued ratio.

19. If corn weighs 56 lb. per bushel, barley 48 lb. per bushel, and oats 32 lb. per bushel, express the three quantities of Prob. 17 in a continued ratio (by weight).

20. Divide 120 into 3 parts that are in the ratio 2:3:5.

21. Divide 240 into 3 parts in the ratio of 3:4:5.

22. A 150-gal. mixture of alcohol and water is to be 8% alcohol (by volume). How much alcohol must be used?

23. How many gallons of alcohol would have to be used in Prob. 22 to make a mixture 8% alcohol by weight? (Compare with Example 7.)

24. In Prob. 22, what is the ratio (by weight) of alcohol to water?
25. In Prob. 23, what is the ratio (by volume) of alcohol to water?

118. Proportion. A proportion is a statement of equality between ratios. Four numbers a, b, c, and d are in proportion if the ratio of the first to the second equals the ratio of the third to the fourth. The proportion may be written $\frac{a}{b} = \frac{c}{d}$, or $a:b = c:d$. The first form is preferable in most cases, but the second must be used when a *continued proportion* is written, *viz.*, $a:b:c = d:e:f$.

In the proportion $\frac{a}{b} = \frac{c}{d}$, the numbers a and d are called the *extremes* and b and c the *means* of the proportion. In any proportion, the product of the extremes equals the product of the means, *i.e.*, $ad = bc$.

If $\frac{a}{b} = \frac{c}{d}$, a and b are said to be *in proportion to* c and d. They are also said to be in *inverse* proportion to d and c; *i.e.*, $\frac{a}{b}$ equals the *inverse* of $\frac{d}{c}$ if $\frac{a}{b} = \frac{c}{d}$.

Example 1. The circumferences of circles are in proportion to their respective diameters. If a circle whose diameter is 7 ft. has a circumference of 44 ft., what is the circumference of a circle whose diameter is 11 ft.?

Solution: Let D_1 and D_2 be the diameters of the respective circles, and let C_1 and C_2 be their circumferences. Then $\frac{C_1}{C_2} = \frac{D_1}{D_2}$, since the circumferences are in proportion to (in the same ratio as) the diameters. Then $\frac{44}{C_2} = \frac{7}{11}$, from which $C_2 \doteq 69$ ft. (check). Note that (in this example) it is not necessary to know the formula for the circumference of a circle. This is the principal advantage of the use of proportion.

Example 2. Express the fact that the volumes of spheres are in proportion to the cubes of their diameters.

Ans.: $\frac{V_1}{V_2} = \frac{D_1^3}{D_2^3}$, where V_1 and V_2 are the volumes of two spheres and D_1 and D_2 are their respective diameters.

EXERCISES AND PROBLEMS (118)

1. Prove that the product of the extremes equals the product of the means, in any proportion.

2. From the proportion $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a}{c} = \frac{b}{d}$.

3. Given $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{d}{b} = \frac{c}{a}$.

4. Form four proportions from the equation $xy = mn$.

5. Find a fourth proportional to 2, 3, and 10; *i.e.*, find the fourth term of a proportion containing 2, 3, and 10 as the first three members.

6. Find a fourth proportional to 4, 7, and 20.

7. In the proportion $\frac{a}{m} = \frac{m}{b}$, m is called the *mean proportional* between a and b. Prove that $m = \pm\sqrt{ab}$.

8. Find a mean proportional between 3 and 48.

9. Find a mean proportional between 2 and 3.

10. A man 6 ft. high measures his shadow and finds it to be 10 ft. long. He measures the shadow cast by a flagpole and finds it to be 80 ft. long. What is the height of the flagpole?

11. When two objects balance on a lever, their distances from the point of support (called the *fulcrum*) are in *inverse* proportion to their weights. Suppose that Charles and Kenneth balance on a teeterboard and that Charles is 8 ft. from the fulcrum. If Charles weighs 140 lb. and Kenneth 160, how far is Kenneth from the fulcrum?

12. If Charles now holds his little brother, who weighs 40 lb., to what position must Kenneth move in order to balance them?

13. Charles and Kenneth balance on a teeterboard 18 ft. long. If each of them sits 1 ft. from the end of the board, how far is each from the fulcrum?

14. The employees of a firm are asked to contribute to the Red Cross. Each contributes in proportion to his salary, and the manager, whose salary is \$400 per month, contributes \$25. How much is contributed by a clerk who receives \$175 per month? How much by a janitor who is paid \$100 per month?

15. The areas of circles are in proportion to the squares of their diameters. If the area of a circle is 50 sq. ft. and its diameter is 7.979 ft., what is the area of a circle whose diameter is 3.15 ft.? Use logarithms in making the final calculation.

16. What is the diameter of a circle whose area is 26 sq. ft.? (Use logarithms.)

17. Under some conditions, the force of air resistance that retards an airplane is proportional to the square of the speed at which it travels. If at 300 m.p.h. the force of retardation is 1,700 lb., what will be the force of retardation at 200 m.p.h.?

18. At what speed would the force of retardation for the airplane of Prob. 17 be 2,500 lb.?

19. The number of oscillations (complete swings back and forth) the pendulum of a clock will make in an hour is *inversely* proportional to the square root of its length. If a pendulum 2 ft. long makes 2,300 oscillations in an hour, how many will be made by a pendulum 8 ft. long?

20. How long must a pendulum be in order to make 3,600 oscillations per hour, or one complete swing per second? (Use logarithms.)

119. Variation. Suppose that $y = kx$, where k is a constant and x and y are variable quantities. Then $\frac{y}{x} = k$, or the ratio of y to x is a constant. Under these conditions, y is said to *vary directly* as x, or to be *directly proportional* to x. (The word *directly* is usually omitted, and y is said to vary as x, or to be proportional to x.) The constant k is called the *constant of proportionality.* No matter how x changes, the ratio of y to x will always be k, if the relation $y = kx$ is satisfied.

Example 1. If y varies as x (or is proportional to x) and $y = 18$ when $x = 3$, find y when $x = 4$.

Solution: Since y varies as x, $y = kx$. Let us first find k, by substituting $y = 18$ and $x = 3$. Then $18 = (k)(3)$, or $k = 6$; therefore $y = 6x$. Then, when x changes to 4, $y = (6)(4) = 24$. This example can also be solved by writing a proportion; for the various values of y are in proportion to the corresponding values of x. Thus, $\frac{y_1}{y_2} = \frac{x_1}{x_2}$, where $y_1 = 18, x_1 = 3, x_2 = 4$; and y_2 is to be found. Then $\frac{18}{y_2} = \frac{3}{4}$, or $y_2 = 24$.

Example 2. The weight of a piece of wire varies as its length. If 55 ft. of copper wire of a certain size weighs 1.1 lb., what is the weight of 75 ft. (of the same wire)?

To say that the weight of a piece of wire *varies* is somewhat confusing. What is actually meant is that the weights of different pieces of the same kind and size of wire are in proportion to their respective lengths; *i.e.*, $\frac{W_1}{W_2} = \frac{L_1}{L_2}$, where W_1 and W_2 are the weights and L_1 and L_2 the lengths of two pieces of the same kind of wire. The same information is conveyed by writing $W = kL$, where k is the constant of proportionality and W and L are *thought of* as *variables.* This means that, if we think of a piece of wire whose length is increasing, the value of its weight changes as its length changes. Thus $W = kL$, always, no matter what value of L is chosen.

Now for the solution: If $W = kL$, and if for one piece of wire $W = 1.1$ lb. and $L = 55$ ft., then $k = \frac{1.1}{55}$ lb. per foot. When L is increased to 75, W will increase to $W = \frac{1.1}{55} \times 75 = 1.5$ lb. Examine Fig. 49, in which this example is illustrated by a graph. Be sure you actually visualize a piece of wire increasing in length from 55 to 75 ft. Do not think of two pieces of wire, but think of one piece of wire that you can vary in size at will. This is a fundamental step in establishing the concept of variation.

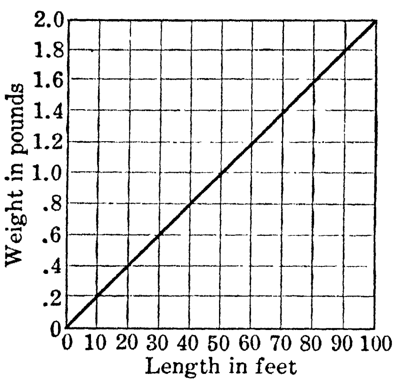


FIG. 49.—Example 2, direct variation.

Example 3. The weight of a piece of wire varies as the square of its diameter or is proportional to the square of its diameter. If a piece of wire .02 in. in diameter weighs 5 oz., how much will another piece .05 in. in diameter (and of the same length) weigh?

Solution: Again the idea of a varying piece of wire, this time varying in diameter, is better than that of two separate pieces of wire. The variation is expressed by $W = kD^2$, where D is the diameter. Using $W = 5$ oz. and $D = .02$ in., we can write $5 = k(.02)^2$, from which $k = \frac{5}{.0004} = 12{,}500$ oz. per square inch. If D is increased to .05, then $W = 12{,}500(.05)^2 = 31.25$ oz., indicating that the weight increases to 31.25 oz. (Be sure to visualize the piece of wire as growing in diameter. Remind yourself that when its diameter is two times as large as at first its weight will be four times as large; when its diameter is three times as large, its weight will be nine times as large; etc. Refer to Fig. 50, in which this example is illustrated by a graph.)

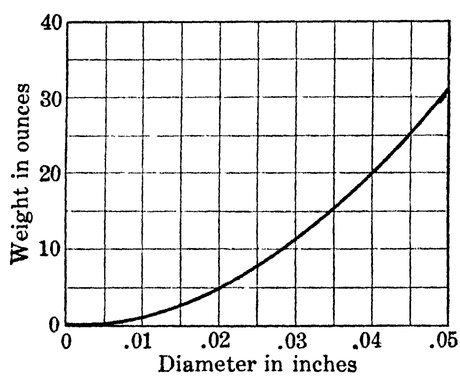


FIG. 50.—Example 3, variation as the square.

In the preceding examples, it was shown that the weight of a piece of wire of given size and material varies as its length and

that a piece of wire of given length and material varies as the square of its diameter. This suggests that the weight of a piece of wire of a given material varies *jointly* as its length *and* the square of its diameter, or as the product of its length and the square of its diameter; therefore $W = kLD^2$.

Example 4. If a piece of wire .0727 in. in diameter and 35.4 ft. long weighs 42.5 oz., what will be the weight of a piece 258 ft. long and .094 in. in diameter?

Solution: $W = kLD^2$, where $W = 42.5$, $L = 35.4$ ft., $D = .072$ in. Then $42.5 = k(35.4)(.072)^2$, or $k = \dfrac{42.5}{(35.4)(.072)^2} \doteq 231.6$. Then, if D is changed to .094 in. and L to 258 ft.,

$$W = kLD^2 = (231.6)(258)(.094)^2 \doteq 528 \text{ oz., or } 33 \text{ lb.}$$

When y varies as $\dfrac{1}{x}$, y is said to vary *inversely* as x, or is *inversely proportional* to x. This means, of course, that $y = k\left(\dfrac{1}{x}\right)$, or $y = \dfrac{k}{x}$. This equation can be rewritten as $xy = k$, or as $x = \dfrac{k}{y}$; hence the conclusion: When the product of two quantities is a constant, each varies inversely as the other. This type of variation is illustrated by the graph in Fig. 51, which is an equilateral hyperbola.

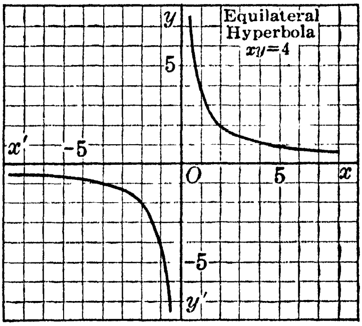


FIG. 51.—Inverse variation.

Many phenomena in nature involve inverse variation. For example, the force of gravitational attraction between two planets, or between a planet and the sun, varies inversely as the square of the distance between them $\left(F = \dfrac{k}{D^2}\right)$. The intensity of illumination produced by a source of light varies inversely as the square of the distance from it; therefore if the earth were twice as far from the sun, it would receive one-fourth as much light. Other physical phenomena involving variation, either direct or inverse, are illustrated in the examples and exercises.

Example 5. The electrical resistance of a piece of wire varies inversely as the square of its diameter. If a piece of wire whose diameter is .055 in.

has a resistance of 22.7 ohms, what is the resistance of another piece of wire (of the same material and length) whose diameter is .073 in.? (NOTE: The *ohm* is the unit of electrical resistance.)

Solution: $R = \frac{k}{D^2}$, where R is the resistance of the wire and D is its diameter. Then

$$22.7 = \frac{k}{(.055)^2}, \qquad \text{or} \qquad k = 22.7(.055)^2 \doteq .06867$$

If the diameter is increased to .073 in., the resistance is

$$R = \frac{k}{D^2} \doteq \frac{.06867}{(.073)^2} \doteq 12.89 \text{ ohms}$$

EXERCISES (119)

1. If a varies as b, and $a = 20$ when $b = 15$, find a when $b = 27$.

2. If y varies as x, and $y = 723$ when $x = 11.7$, find y when $x = 27.8$. (Use logarithms. After you obtain $\log k$, do not evaluate k, for you will need only $\log k$.)

3. If m varies as n, and $m = 2{,}750$ when $n = 357$, find m when $n = 95.2$.

4. If y varies as x^2, and $y = 189$ when $x = 3$, find y when $x = 4$.

5. If m varies as $2n - 3$, and $m = 33$ when $n = 7$, find m when $n = 16$.

6. If y varies *inversely* as x, and $y = 25$ when $x = 15$, find y when $x = 5$.

7. If y varies inversely as x, and $y = 27$ when $x = 2$, find y when $x = 3$.

8. If 50 ft. of copper wire of a certain size weighs .252 lb., what is the weight of 175 ft.?

9. If a piece of wire .0234 in. in diameter weighs 1.35 lb., what is the weight of an equal length of wire .0327 in. in diameter?

10. The electrical resistance of a piece of wire varies as its length. If 10 ft. of a certain kind of wire has a resistance of 42 ohms, what will be the resistance of 17 ft.?

11. The distance a freely falling object falls in a given time (after being dropped) varies as the square of the time it is allowed to fall. If it is observed that an object falls 64 ft. in the first 2 sec., how far does it fall in the first 5 sec.?

12. The height to which a ball can be thrown straight upward varies as the square of the velocity with which it is thrown. If a ball thrown upward at 16 ft. per second rises 4 ft. in the air, to what height will the ball rise if it is thrown at 48 ft. per second?

13. If a bomb is dropped from an airplane, the time required for it to fall would vary as the square root of the height from which it falls, if it were not for the effect of air resistance, which makes it take a longer time. If a bomb dropped from 7,200 ft. reaches the ground in 15 sec., how long would it take to fall from 30,000 ft., neglecting the effect of air resistance?

14. If a ball is batted into the air at an elevation of 45°, the distance at which it hits the ground varies as the square of the velocity with which it leaves the bat, if air resistance is neglected. If a ball leaves the bat at 64 ft. per second and hits the ground 256 ft. away, what velocity is needed to send it 400 ft. for a "home run"?

15. The volume of a cylinder varies as its height and as the square of its diameter. If the volume of a cylinder 22 ft. long and 12.7 in. in diameter is 19.35 cu. ft., what is the volume of a cylinder 27.5 ft. long and 18.3 in. in diameter?

16. If a piece of wire .0125 in. in diameter has an electrical resistance of 7.25 ohms, what is the resistance of an equal length of wire whose diameter is .063 in.? (*Cf.* Example 5.)

17. If 42 ft. of wire .027 in. in diameter has a resistance of 17.4 ohms, what will be the resistance of 32 ft. of wire .045 in. in diameter?

18. The volume of an air bubble increases as it rises from the bottom of the sea. Its volume varies *inversely* as $14.7 + .445D$, where D is the distance (in feet) of the air bubble below the surface. If a bubble contains .25 cu. in. of air when it is 1 mile (5,280 ft.) below the surface, what will be its volume when it reaches the surface ($D = 0$)?

REVIEW QUESTIONS

1. If a ratio involves only numbers, what is meant by the ratio of zinc to copper in an alloy composed of these two metals?

2. What is wrong with the statement, "A mixture consists of 10% alcohol and 90% water"?

3. Show that the statement, "y varies as x," is equivalent to a proportion.

4. List four examples of variation not mentioned in this chapter, such as, "The amount of gasoline used on a trip varies as the number of miles traveled."

5. Represent each of the examples in Question 4 in the form of a proportion, such as $\frac{a}{b} = \frac{c}{d}$ or $\frac{y_1}{y_2} = \frac{x_1}{x_2}$, indicating the significance of each symbol.

CHAPTER 18

TRIGONOMETRY

Some of the most important applications of ratio and proportion occur in connection with *trigonometry*, one of the oldest and most practical branches of mathematics. Trigonometry deals with the relationships among the sides and angles of triangles, relationships that are made use of in surveying, astronomy, architecture, and military science—in fact, in all branches of engineering and physical science.

120. A Problem in Surveying. In Fig. 52, the point C is on the shore of a lake and the point B is on a large island about half a

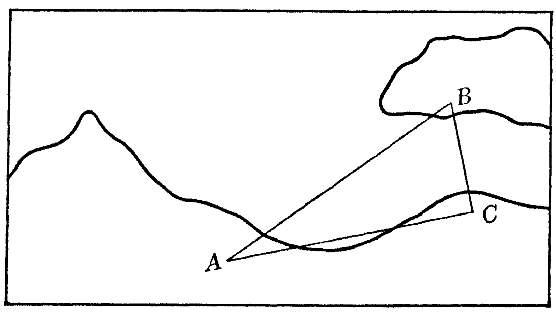


FIG. 52.—A problem in surveying.

mile from shore. A bridge is to be built between B and C; therefore a very precise measurement of the distance BC is necessary.

The surveyor sets up his transit* at point C and sights through it toward B to obtain a reference position. Next, he turns the transit to a position at right angles to the line BC and directs his helper in locating a marker at A, in line with the 90° position. Now, moving to the point A, he sights toward C for a reference position, then toward B, making an accurate measurement of the angle at A, which he records as exactly 24°. Finally, the distance AC is carefully measured as 5,124 ft., accurate to within a few inches. (Reread the problem at this point, making sure that you have the situation well in mind.)

* A surveyor's transit is a small telescope designed for making precise measurements of angles.

In order to determine the distance BC, one might make an accurate scale drawing of the triangle ABC. This would yield a value of the distance BC accurate to within about 20 ft., an accuracy of about 1% being assumed in constructing and reading the drawing.

A more accurate method of determining the distance BC is to make use of a table listing the ratios of the sides of right triangles. The values of the ratio $\frac{BC}{AC}$ for right triangles in which the angle A has any of the values 1°, 2°, 3°, . . . 90° are listed in Table 3, beginning on page 290, in the column headed "tan." Referring to the table, find the angle 24° in the column at the left. Now shift to the column headed "tan," and there read .4452, which is the desired value of the ratio $\frac{BC}{AC}$. Since AC = 5,124 ft. and $\frac{BC}{AC} \doteq .4452$,

$$BC \doteq .4452 \times 5{,}124 \doteq 2{,}281 \text{ ft}$$

This result is accurate to within about 6 in.

121. Trigonometric Ratios. In Fig. 53, suppose that a vertical line BC can be moved from left to right at will, taking on any of the positions similar to $B'C''$, $B''C''$, etc., so that the triangle ABC can be thought of as variable in size. Use the diagram to verify the fact that $\frac{BC}{AC} = \frac{2}{3}$ for each of the positions shown.** The ratio $\frac{BC}{AC}$ is thus independent of the size of the right triangle so long as the angle at A has the value shown. Likewise, the ratios $\frac{BC}{AB}$ and $\frac{AC}{AB}$ are constant for a given angle A. That these ratios are constant can be proved by means of the following theorem:

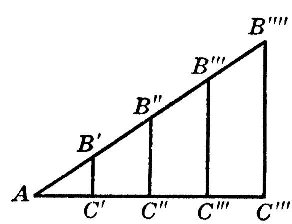


FIG. 53.—Trigonometric ratios.

The corresponding sides of similar triangles are in proportion.

($AB'C'$, $AB''C''$, etc., are similar triangles, because they have the same angles.)

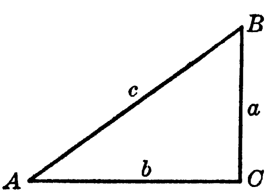


FIG. 54.—Standard right triangle.

The sides of a triangle are usually represented by small letters, corresponding in each

case to the angle opposite the side, as in Fig. 54. Note that a is the side opposite angle A, etc.

Because the ratios of the sides of a right triangle depend only on (either of) its acute angles, these ratios are called *functions* of the angles and are given individual names, as follows:

The *sine* of angle A is $\frac{a}{c}$, or $\sin A = \frac{a}{c}$

The *cosine* of angle A is $\frac{b}{c}$, or $\cos A = \frac{b}{c}$

The *tangent* of angle A is $\frac{a}{b}$, or $\tan A = \frac{a}{b}$

Expressed in words, for any acute angle A,

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \cos A = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}}$$

These three relations should be memorized.**

The values of the sine, cosine, and tangent, for various angles from 0 to 90°, are listed in Table 3, beginning on page 290. (Their values are rounded off at four figures in this table.) The use of the table will be illustrated by examples.

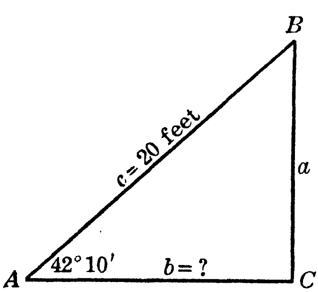


Fig. 55.—Example 1.

Example 1. In the right triangle of Fig. 55, determine the length of the side b.

Solutions: In Fig. 55, $c = 20$ ft. and $A = 42°10'$. The relation involving A, c, and b is

$$\cos A = \frac{b}{c}, \qquad \text{from which} \qquad b = c \cos A$$

From the table, $\cos 42°10' \doteq .7412$; hence $b \doteq 20 \times .7412 \doteq 14.82$ ft.

NOTE: For angles between 0 and 45°, read the angles in the left columns, and use the column designations sin, tan, cos, *at the top.*

Example 2. In a right triangle, $c = 32$ ft. and angle B is $74°20'$. Determine b.

Solution: $\frac{b}{c} = \sin B$; hence $b = c \sin B$. Draw a right triangle,

label its sides, and verify that $\frac{b}{c} = \cos A = \sin B$, since $\frac{b}{c}$ is

$$\frac{\text{the side opposite } B}{\text{hypotenuse}}$$

To find the functions of angles between 45 and 90°, it is necessary to read the angles in the *right* columns and use the column designations cos, tan, sin, *at the bottom.* Now refer to the table, and find

$$\sin 74°20' = .9628.$$

Then

$$b = c \sin B = 32 \times .9628 \doteq 30.81 \text{ ft}$$

CAUTION: When the angle is read in the *right* column, the column designations at the *bottom* must be used. In this example, the use of the column headed *sin* at the top would produce the incorrect result .2700, which is cos 74°20′, or sin 15°40′.

Example 3. In a right triangle, $a = 5$ ft. and $b = 12$ ft. Determine the angles of the triangle.

Solution: (Draw a triangle for use in following the solution.) Since angle C is 90°, and since the sum of the three angles in any triangle is 180°, the sum of A and B is always 90° in a right triangle. This means that it is necessary to find only one (either) of the acute angles. Now $\frac{a}{b} = \tan A$; hence (in this triangle) $\tan A = \frac{5}{12} \doteq .4017$. Referring to the *tan* column in the table of trigonometric functions, we find that for $\tan A \doteq .4017$ angle A is between 21°50′ and 22°. For the present, we shall take the nearest listed value, or 21°50′. (Interpolation will be considered in the next section.) Then $A \doteq 21°50'$, and

$$B \doteq 90° - 21°50' = 68°10'.$$

EXERCISES (121)

In the right triangles described by the following data, determine the values indicated by question marks. Determine the values of the sides to four figures, but evaluate the angles only to the nearest value given in the table, in order to avoid interpolation. The multiplications and divisions may be performed by means of logarithms or by arithmetic. Choose the method best suited to the specific calculation. *Begin each exercise by drawing a triangle* and indicating the numerical values known, in order to prevent errors.

	a	*b*	*c*	*A*	*B*
1.	?	?	50 ft.	37°20′	?
2.	?	?	40 in.	72°40′	?
3.	?	200 ft.	?	?	41°10′
4.	?	140 ft.	?	?	63°50′
5.	?	?	25 ft.	?	17°30′
6.	?	?	30 in.	?	23°20′
7.	20.0 ft.	?	?	31°50′	?
8.	?	60 ft.	?	?	77°
9.	....	44.7 ft.	100 ft.	?	?
10.	....	31.6 ft.	50 ft.	?	?
11.	24.2 ft.	40 ft.	....	?	?
12.	14.7 ft.	300 ft.	....	?	?
13.	....	3.25 in.	57.4 in.	?	?
14.	....	14.6 ft.	24.7 ft.	?	?
15.	?	?	47.4 ft.	?	32°
16.	242 ft.	?	?	79°10′	?
17.	?	5,650 ft.	?	17°20′	?
18.	447 ft.	2.17 ft.	....	?	?

122. Interpolation. The method of interpolation used with trigonometric tables is the same as that used with the table of logarithms. There are two fundamental problems:

1. Given an acute angle not listed in the table, to find the corresponding sine, cosine, or tangent.

2. Given a sine, cosine, or tangent not listed in the table, to find the corresponding acute angle.

The first case will be considered in this section.

Example 1. Find sin 43°13′.

Solution: From the table, $\sin 43°10' \doteq .6841$

$\sin 43°20' \doteq .6862$

The tabular difference is 21. Now, 43°13′ is $\frac{3}{10}$ of the way from 43°10′ to 43°20′; hence the amount to be added is $.3 \times 21 \doteq 6$ (mentally). Then $6841 + 6 = 6847$, and $\sin 43°13' \doteq .6847$. Only the result should be written.

Example 2. Find cos 37°23′.

Solution: $\cos 37°20' \doteq .7951$

$\cos 37°30' \doteq .7934$

The tabular difference is 17, and 23′ is $\frac{3}{10}$ of the way from 20′ to 30′. Thus, the amount $.3 \times 17 \doteq 5$ is to be *subtracted* from 7951, giving

7946. Then $\cos 37°23' \doteq .7946$. NOTE: In order to decide whether the correction is to be added to the smaller listed value or subtracted from the larger one, always check to see whether you are interpolating "upward" or "downward." In this case, the result is to be $\frac{3}{10}$ of the way *from* 7951 *to* 7934; hence the correction is subtracted from 7951.

Example 3. Find $\tan 31°07'$.

Solution:
$$\tan 31°00' \doteq .6009$$
$$\tan 31°10' \doteq .6048$$

The tabular difference is 39; hence the correction is $\frac{7}{10} \times 40 \doteq 28$, and $\tan 31°07' \doteq .6037$.

Example 4. Find $\cos 78°24'$.

Solution:
$$\cos 78°20' \doteq .2022$$
$$\cos 78°30' \doteq .1994$$

The tabular difference is 28, and $.4 \times 28 \doteq 11$ (compute mentally, $28 = 20 + 8$, $.4 \times 20 = 8$, $.4 \times 8 = 3.2$, total is 11). Then

$$2022 - 11 = 2011,$$

and $\cos 78°24' \doteq .2011$. Nothing need be written except the result .2011. Note that 11 is subtracted, since the desired value is $\frac{4}{10}$ of the way *from* .2022 *to* .1994.

EXERCISES (122)

Evaluate the following:

1. $\sin 78°10'$	2. $\sin 28°50'$	3. $\tan 42°12'$
4. $\tan 17°48'$	5. $\sin 63°25'$	6. $\sin 6°40'$
7. $\tan 62°32'$	8. $\tan 12°45'$	9. $\sin 36°13'$
10. $\sin 4°17'$	11. $\cos 14°20'$	12. $\cos 27°40'$
13. $\tan 62°45'$	14. $\cos 51°12'$	15. $\cos 79°15'$
16. $\cos 72°06'$	17. $\tan 47°17'$	18. $\tan 64°23'$
19. $\tan 48°21'$	20. $\cos 27°48'$	21. $\sin 21°53'$
22. $\cos 87°53'$	23. $\tan 31°47'$	24. $\tan 81°38'$

123. Finding the Angle. In finding the angle that corresponds to a value of sine, cosine, or tan not listed in the table, apply the method of the preceding section, in reverse.

Example 1. Find the angle A, if $\sin A = .4258$.
Solution: The nearest listed values are

$$\sin 25°10' \doteq .4253$$
$$\sin 25°20' \doteq .4279$$

The tabular difference is 26, and 4258 exceeds 4253 by 5; hence the desired angle exceeds 25°10′ by $\frac{5}{26} \times 10' \doteq 2'$. Then $A \doteq 25°12'$. Always carry the values only to the nearest minute of angle.

Example 2. If $\cos A = .6976$, find A.

Solution:

$$\cos 45°40' \doteq .6988$$
$$\cos 45°50' \doteq .6967$$

The tabular difference is 21, and 6976 is 12 *less* than 6988. (Note that this procedure is necessary in order to find how much angle A exceeds 45°40′.) Then the amount to be added is $\frac{12}{21} \times 10' \doteq 6'$, and $A \doteq 45°46'$. Check this example carefully, making sure you understand why one determines the amount $6988 - 6976 = 12$, rather than $6976 - 6967 = 9$, in interpolating.

Example 3. If $\tan A = .2174$, find A.

Solution:

$$\tan 12°10' \doteq .2156$$
$$\tan 12°20' \doteq .2186$$

The tabular difference is 30, and 2174 exceeds 2156 by 18; hence the amount to add is $\frac{18}{30} \times 10' = 6'$. Then $A \doteq 12°16'$.

Example 4. If $\tan A = 21.89$, find A.

Solution:

$$\tan 87°20' \doteq 21.47$$
$$\tan 87°30' \doteq 22.90$$

The tabular difference is 143, and 2189 exceeds 2147 by 42; therefore the amount to add is $\frac{42}{143} \times 10' \doteq 3'$. Then $A \doteq 87°23'$. Note that $\frac{42}{143} \times 10'$ can be evaluated mentally; for only the nearest minute of angle is to be found.

EXERCISES (123)

Determine the angle:

1. $\sin A = .1429$	2. $\sin B = .3184$	3. $\sin A = .3726$
4. $\sin A = .3185$	5. $\sin B = .8143$	6. $\sin B = .9642$
7. $\cos A = .8263$	8. $\cos A = .8146$	9. $\cos B = .2178$
10. $\cos B = .3064$	11. $\cos A = .9164$	12. $\cos B = .1248$
13. $\tan A = .2416$	14. $\tan A = .3142$	15. $\tan A = .4868$
16. $\tan A = .3417$	17. $\tan A = 1.814$	18. $\tan A = 1.778$
19. $\tan A = 1.444$	20. $\cos A = .3127$	21. $\tan A = .8145$
22. $\cos A = .0125$	23. $\tan A = 1.013$	24. $\sin A = .0041$

124. Logarithms of the Functions. Consider the following problem:

Example 1. In a right triangle, $A = 42°17'$ and $c = 24.40$ ft. Find a and b. (Always draw a right triangle, and label the sides with the known values.)

Solution: $\frac{a}{c} = \sin A$, where c and A are known. Then $a = c \sin A$. In order to solve by logarithms, we write: $\log a = \log c + \log (\sin A)$. Now, we can look up $\sin A$ in a table of trigonometric functions and then use a table of logarithms to determine the logarithm of $\sin A$. It is more convenient, however, to obtain $\log \sin A$ *directly* from a special table listing the *logarithms* of the trigonometric functions. For convenience, the logarithms of the trigonometric functions are listed in Table 3, page 290, with each logarithm next to the corresponding value of the function.

The numerical values of sine and cosine are always smaller than 1; hence the logarithms of sine and cosine always have negative characteristics. Similarly, the tangent is smaller than 1 for angles between 0° and 45°, so that in this range the logarithm of the tangent also has a negative characteristic. For example: $\sin 42° \doteq .6691$, $\log \sin 42° \doteq .8255 - 1$; $\tan 23° \doteq .4245$, $\log \tan 23° \doteq .6279 - 1$. In order to save space in the table, the characteristic is increased by 10 when it is negative. Thus, $.8255 - 1$ is written in the table as 9.8255, while $.8255 - 3$ would be listed as 7.8255, etc. When the characteristic given in the table is 7, 8, or 9, it must be remembered that the actual characteristic is negative, and should be recorded (as negative) in the usual fashion. For example, refer to the table to observe that $\log \sin 3°$ is listed in the table as 8.7188, whereas it actually equals $.7188 - 2$. Check also $\log \tan 13° = .3634 - 1$, and $\log \cos 84° = .0192 - 1$. Always change the characteristic mentally, by subtracting 10, and record the logarithm in the form $.0192 - 1$.

The use of the table will be illustrated by examples. First, let us complete Example 1, in which $A = 42°17'$ and $c = 244.0$ feet, with the values of a and b to be found.

Solution: $\frac{a}{c} = \sin A$; hence $a = c \sin A$.

Then

$$\begin{aligned} \log a &= \log c + \log \sin A \\ \log c &= \log 2_\wedge 44 \doteq 2.3874 \\ \underline{\log \sin A = \log \sin 42°17'} &\underline{\doteq .8279 - 1} \text{ (check this)} \\ \log a &\doteq 2.2153 \\ a \doteq \text{antilog } 2.2153 &\doteq 1_\wedge 64.2 \text{ ft.} \end{aligned}$$

Likewise $\frac{b}{c} = \cos A$ or $b = c \cos A$

Then
$$\begin{aligned} \log b &= \log c + \log \cos A \\ \log c &\doteq 2.3874 \text{ from above} \\ \log \cos 42°17' &\doteq .8691 - 1 \\ \hline \log b &\doteq 2.2565 \\ b &\doteq \text{antilog } 2.2565 \doteq 1_\wedge 80.5 \text{ ft.} \end{aligned}$$

Example 2. If $A = 67°42'$ and $b = 141$ ft., find c and B. (Draw a triangle.)

Solution: $\frac{b}{c} = \cos A$; hence $c = \frac{b}{\cos A}$, or

$$\log c = \log b - \log \cos A$$

Now $\log b = \log 1_\wedge 41 \doteq 2.1492$

and $\log \cos A = \log \cos 67°42' \doteq .5792 - 1$ (check this)

subtracting, $\log b \doteq 2.5700$

Then $c \doteq \text{antilog } 2.5700 \doteq 3_\wedge 71.5$ ft.

In order to determine B, we use $\frac{b}{c} = \sin B$, from which

$$\log \sin B = \log b - \log c.$$

From above,
$$\begin{aligned} \log b &\doteq 2.1492 \\ \log c &\doteq 2.5700 \\ \hline \log \sin B &\doteq .5792 - 1 \end{aligned}$$

We now locate 9.5792 in the table (log sin column), finding

$$B \doteq 22°18' \text{ (check this)}$$

EXERCISES AND PROBLEMS (124)

Evaluate:

1. $\log \sin 68°12'$
2. $\log \sin 17°53'$
3. $\log \tan 74°32'$
4. $\log \tan 23°42'$
5. $\log \cos 41°26'$
6. $\log \cos 34°47'$
7. $\log \cos 77°13'$
8. $\log \tan 44°17'$
9. $\log \sin 4°43'$
10. $\log \tan 87°18'$

Find the angle:

11. $\log \sin A = .3947 - 1$
12. $\log \sin A = .5724 - 1$
13. $\log \cos A = .8135 - 1$
14. $\log \cos A = .9080 - 1$
15. $\log \tan A = .8132 - 1$
16. $\log \tan A = .7800 - 1$
17. $\log \tan A = 0.5600$
18. $\log \tan A = 0.2165$
19. $\log \sin A = .8815 - 2$
20. $\log \tan A = .7860 - 2$

Determine the missing parts of the following right triangles:

	a	b	c	A	B
21.	?	?	575 ft.	37°34′	?
22.	?	?	24.7 in.	72°47′	?
23.	?	34.2 ft.	?	?	41°17′
24.	?	147 ft.	?	?	63°55′
25.	?	?	25.7 ft.	?	17°33′
26.	?	?	37.2 in.	?	23°20′
27.	27.9 ft.	?	?	17°27′	?
28.	?	67.2 ft.	?	?	77°54′
29.	?	44.7 ft.	227 ft.	?	?
30.	?	51.6 ft.	57.2 ft.	?	?
31.	24.2 ft.	175 ft.	?	?	?
32.	14.7 ft.	312 ft.	?	?	?

33. If a flagpole 26.7 ft. tall produces a shadow 56.3 ft. long, what is the angle of elevation of the sun? (In Fig. 56, the angle CAB is called the angle of elevation of B as viewed from A. The angle DBA is called the angle of depression of A as viewed from B.)

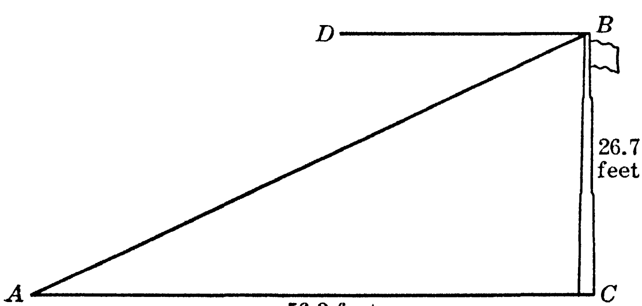


Fig. 56.—Angle of elevation.

34. A ladder 17.5 ft. long leans against a building. If it makes an angle of 58°43′ with the ground, at what height on the building does it touch?

35. From an airplane at a height of 15,500 ft., the angle of depression of a submarine is 12°17′. At 250 m.p.h., how long will it take the airplane to reach a spot directly over the submarine?

36. The altitude of an isosceles triangle is 37.5 ft., and the angle at the top is 62°44′. How long is its base?

The remaining problems in this list require careful analysis, since they involve other principles in addition to those of trigonometry. Draw a diagram of each problem, and study it until you understand the situ-

ation well enough to apply the principles learned in this chapter. Do not be discouraged if you find these problems difficult.

37. A shell from an antiaircraft gun requires 22 sec. to reach a spot 25,000 ft. directly above the gun. An enemy plane at an altitude of 25,000 ft. is hit as it passes directly overhead, at 300 m.p.h. What was the angle of elevation of the plane (as seen from the position of the gun) at the instant the gun was fired?

38. A bomber flies overhead at 325 m.p.h. If its altitude is 27,300 ft., what will be its angle of elevation in 3 min.? in 5 min.?

39. An airplane flies on a direct line from Memphis to Chicago. If its air speed is 255 m.p.h., and if there is a 50-m.p.h. wind at right angles to the path of the plane, how many degrees off course must the plane be directed in order to fly on course?

40. In Prob. 39, what will be the ground speed of the plane?

41. In a right triangle ABC, the point C represents the position of an enemy torpedo plane "picked up" by radar from a carrier at A, 100 miles west of C. The enemy plane is flying due north at 200 m.p.h. Fighter planes dispatched from the carrier fly in a straight line to B, where they intercept the enemy plane. If the angle at A is 47°, at what speed did the planes travel?

42. If the fighter planes had been able to travel 25 m.p.h. faster, at what angle (at A) should they have flown in order to make the interception?

43. Two planes circling over a carrier are dispatched simultaneously, one heading north at 350 m.p.h. the other east at 295 m.p.h., while the carrier remains stationary. After 15 min., what is the distance between the two planes?

44. In Prob. 43, if at the end of the 15 min. the carrier started moving west at 30 m.p.h. and at the same time the first plane started to return to the carrier, how many degrees west of south would the plane have to fly in order just to meet the carrier?

45. In Prob. 44, how much time would be required for the plane to return?

46. Suppose that a surveyor is making a map of an uncharted coast line and that he has located the points X and Y 5 miles apart, both on the shore. He plans to use the line (XY) between them as a base line on his map. The point Z is a prominent landmark on one end of an island several miles off the coast. The surveyor wishes to locate point Z on his map so that he can sketch the outline of the island in its proper position with respect to the coast line.

From point X the surveyor sights through his transit toward the point y to determine the position of the reference line XY, then he turns

the transit and sights through it at Z, measuring the angle at X as 63°24′. Next, he moves to point Y and makes a similar observation, finding the angle at Y to be 47°18′.

Draw a diagram of the problem, and determine the perpendicular distance from Z to the nearest point W on the shore line between X and Y. This problem will require careful analysis and some *original thinking.* Note that the perpendicular ZW divides the original triangle into two right triangles.

CHAPTER 19

EXTRANEOUS ROOTS AND IRRATIONAL EQUATIONS

125. Poor Elmer! This is the story of Elmer, who had a two-dollar bill. When George saw Elmer's two-dollar bill, he said,

"Don't you know that's no better than a one-dollar bill?"

Elmer answered right up, "Why, sure, it's better than a one-dollar bill—it's worth two one-dollar bills."

Said George, "It's only one bill, isn't it?", and Elmer agreed. "It's only one bill, same amount of paper, same amount of ink, same as a one-dollar bill. Here! I'll show you. Let $x = 2$, for the two-dollar bill."

Now Elmer began to be interested.

"Multiply each side of the equation by x," said George, "and you have $x^2 = 2x$."

"Right," said Elmer.

"Next," said George, "subtract 4 from each side, obtaining $x^2 - 4 = 2x - 4$."

"Right so far," said Elmer.

"Then factor each side, giving $(x - 2)(x + 2) = 2(x - 2)$, and divide each side by $x - 2$, obtaining $x + 2 = 2$."

"That was pretty fast," said Elmer, "but I don't see any mistakes. Just what are you up to?"

"Now we have the equation $x + 2 = 2$, and we started with $x = 2$. Adding these two equations,

$$2x + 2 = 4, \text{ from which } x = 1.$$

But x represents the value of your two-dollar bill. Same amount of paper and ink as a one-dollar bill, same value!"

Poor Elmer! He couldn't see a thing wrong with George's algebra; so he traded his two-dollar bill for George's one-dollar bill. Poor Elmer!

The next day, George saw Elmer again and called to him, "Hey, Elmer! Wait a minute! Didn't I show you yesterday that $x + 2 = 2$?"

"Yes," said Elmer.

"Then," said George, "subtract 2 from each side, and what do you have?"

"You have $x = 0$," said Elmer.

"That's right," said George, "so your two-dollar bill was worth nothing! Give me back my one-dollar bill."

Poor Elmer gave George the one-dollar bill; then he happened to think. "Say," he said, "where's my two-dollar bill?" "Oh, that!" said George, "I threw it away. Forget it, it's worth nothing."

Elmer loved algebra. Poor Elmer!

126. Extraneous Roots. To multiply (or divide) each member of an equation by a variable or an expression involving a variable does not produce an equivalent equation, since it introduces (or removes) the roots obtained by equating the multiplier (or divisor) to zero.

Example 1. Suppose $$x = 2 \tag{1}$$

(This equation has only one root.)

$$\text{M}(x), \qquad x^2 = 2x \tag{2}$$

Note that (2) is satisfied by either $x = 2$ or $x = 0$; therefore the *extraneous* root $x = 0$ has been introduced by multiplying each side of the equation by x. Now, subtract 4 from each side of (2),

obtaining $$x^2 - 4 = 2x - 4$$
Factoring, $$(x - 2)(x + 2) = 2(x - 2)$$

This equation is satisfied by $x = 2$ or $x = 0$. If, however, both sides are divided by $x - 2$, leaving $x + 2 = 2$, the equation is satisfied only by $x = 0$, so that the root $x = 2$ has been removed. Since roots may be introduced or removed at will by this method, it is not generally permissible (in solving an equation) to multiply or divide through the equation by an expression *that involves the variable.*

Exception: When, in order to clear an equation of *fractions*, it is necessary to multiply by an expression involving a variable, this does not always introduce an extraneous root:

Example 2. $$1 + \frac{7}{x - 5} = \frac{4}{x - 5}$$

$$\text{M}(x - 5), \qquad x - 5 + 7 = 4$$
$$x = 2$$

In this case, multiplication by $x - 5$ does not cause $x - 5$ to appear as a factor of each member of the equation; hence it does not introduce the extraneous root $x - 5$. Likewise, when an expression such as $x - 5$ does not appear as a factor of both members of the equation, *division* by $x - 5$ does not remove the root $x = 5$, for $x = 5$ is *not* a root of the original equation. One cannot remove a "root that isn't there."

Example 3. $$2x - 7 = 9$$

$$\text{D}(x - 5), \qquad \frac{2x - 7}{x - 5} = \frac{9}{x - 5}$$

The only root of the original equation is $x = 8$. In order to solve the second equation, it will be necessary to multiply by $x - 5$. This reproduces the original equation; hence no root has been removed.

Example 4. Consider the equation $x = -1$

Squaring both sides, $x^2 = 1$

But this equation has two roots, $x = -1$ and $x = 1$; hence the extraneous root $x = 1$ has been introduced by squaring both sides of the equation. After the equation $x = -1$ is squared, one cannot tell whether the result $x^2 = 1$ was obtained from $x = -1$ or $x = +1$.

Example 5. Consider $x + 3 = 4$ (which is equivalent to $x = 1$)

Squaring, $(x + 3)^2 = 16$

This equation could be obtained by squaring any one of the following

$$x + 3 = 4 \tag{1}$$
$$x + 3 = -4 \tag{2}$$
$$-(x + 3) = 4 \tag{3}$$
$$-(x + 3) = -4 \tag{4}$$

Of these, (1) and (4) are equivalent, yielding $x = 1$, and (2) and (3) are equivalent, yielding $x = -7$. That is, in taking the square root, one needs to place the $\pm$ sign on only one side. Then, taking the square root of $(x + 3)^2 = 16$, one has $x + 3 = \pm 4$, or $x = 1$, $x = -7$; so the extraneous root $x = -7$ was introduced by squaring both sides of the original equation $x + 3 = 4$.

127. Irrational Equations. (Optional). Sometimes it is necessary to square both sides of an equation before it can be solved. This is true of *irrational* equations (in one unknown), which are equations in which the unknown appears under a radical sign or

has a fractional exponent, as in the following: $5x + \sqrt{x} = 0$, $x^{\frac{1}{6}} + x^{\frac{1}{3}} + 6 = 0$, $\sqrt{y - 1} + \sqrt{3y} = 3$. In solving such equations, it is necessary to check every root by substitution in the original equation, in order to discover and reject all extraneous roots.

Example 1. $$\sqrt{x + 2} = x - 4$$

Squaring, $$x + 2 = x^2 - 8x + 16$$

Collecting, $$0 = x^2 - 9x + 14$$

Factoring, $$(x - 2)(x - 7) = 0$$

and the roots (?) are $x = 2$ and $x = 7$ by inspection. Substituting 7 for x in the original equation,

$$\sqrt{7 + 2} = 7 - 4 \qquad \text{or} \qquad \sqrt{9} = 3, \text{ which checks}$$

Therefore $x = 7$ is a root of the equation. Substituting 2 for x,

$$\sqrt{2 + 2} = 2 - 4 \qquad \text{or} \qquad \sqrt{4} = -2$$

This does not check; hence $x = 2$ is an extraneous root, not a solution of the original equation.

In solving an irrational equation, a result should never be called a root or solution until it is determined (by checking) that it satisfies the original equation.

Example 2. $$\sqrt{27 - x} + x + 3 = 0$$

Rearranging, $$x + 3 = -\sqrt{27 - x}$$

Squaring, $$x^2 + 6x + 9 = 27 - x$$

(Note that the same equation would have been obtained if the original equation had been $\sqrt{27 - x} - x - 3 = 0$.)

Collecting, $$x^2 + 7x - 18 = 0$$

Factoring, $$(x + 9)(x - 2) = 0$$

Then $x = -9$ and $x = +2$ are *possible* roots.

Checking $x = -9$,

$$\sqrt{27 - (-9)} + (-9) + 3 = 0$$
$$\sqrt{36} - 6 = 0$$
$$6 - 6 = 0, \text{ so } x = -9 \text{ is a root}$$

Checking $x = 2$,

$$\sqrt{27 - 2} + 2 + 3 = 0$$
$$\sqrt{25} + 5 = 0$$
$$5 + 5 = 0, \text{ does not check}$$

hence $x = 2$ is an extraneous root.

An equation may have no roots at all:

Example 3.

$$\begin{aligned}
& & \sqrt{x+4} + 1 &= 0 \\
&\text{T(1),} & \sqrt{x+4} &= -1 \\
&\text{Squaring,} & x + 4 &= 1 \\
&\text{T(4),} & x &= -3 \\
&\text{Checking,} & \sqrt{-3+4} + 1 &= 0 \\
& & \sqrt{1} + 1 &= 0 \\
& & 1 + 1 &= 0, \text{ does not check}
\end{aligned}$$

and $x = 3$ is an extraneous root. In the original equation

$$\sqrt{x+4} + 1 = 0$$

it can be seen that $\sqrt{x+4}$ is positive, since the radical sign indicates the positive square root. Then, since the sum of two positive numbers cannot be zero, $\sqrt{x+4} + 1$ cannot equal zero. Thus $\sqrt{x+4} + 1 = 0$ is a false equation and has *no roots.*

When an irrational equation contains two or more radicals, more steps are necessary to complete the solution:

Example 4.

$$\begin{aligned}
& & 2 + \sqrt{2x} &= \sqrt{x-2} \\
&\text{Squaring,} & 4 + 4\sqrt{2x} + 2x &= x - 2 \\
&\text{Rearranging,} & x + 6 &= -4\sqrt{2x} \\
&\text{Squaring,} & x^2 + 12x + 36 &= 16(2x) \\
& & x^2 - 20x + 36 &= 0 \\
& & (x - 18)(x - 2) &= 0 \\
& & x &= 18 \text{ or } 2 \\
&\text{Checking } x = 18, & 2 + \sqrt{36} &= \sqrt{16} \\
& & 2 + 6 &= 4, \text{ does not check}
\end{aligned}$$

hence $x = 18$ is an extraneous root.

$$\begin{aligned}
&\text{Checking } x = 2, & 2 + \sqrt{4} &= \sqrt{2-2} \\
& & 2 + 2 &= 0, \text{ does not check}
\end{aligned}$$

and $x = 2$ is also an extraneous root.

Example 5. $$2 - \sqrt{2x} = \sqrt{x-2}$$

Note the slight difference between this and Example 4.

$$\begin{aligned}
&\text{Squaring,} & 4 - 4\sqrt{2x} + 2x &= x - 2 \\
&\text{Rearranging,} & x + 6 &= 4\sqrt{2x} \\
&\text{Squaring,} & x^3 + 12x + 36 &= 16(2x) \\
& & x^2 - 20x + 36 &= 0 \\
& & (x - 18)(x - 2) &= 0 \\
& & x = 18 \text{ and } x &= 2 \text{ are possible roots}
\end{aligned}$$

$$
\begin{aligned}
&\text{Checking } x = 18, \quad & 2 - \sqrt{36} &= \sqrt{16} \\
& & 2 - 6 &= 4 \text{ does not check} \\
&\text{hence} & x &= 18 \text{ is an extraneous root} \\
&\text{Checking } x = 2, & 2 - \sqrt{4} &= \sqrt{0} \\
& & 2 - 2 &\equiv 0, \text{ checks} \\
&\text{hence} & x &= 2 \text{ is a root}
\end{aligned}
$$

In solving an irrational equation:

1. Arrange the terms so that the least simple radical expression is alone on one side of the equation.

2. Square (or cube, if the radical is cubic) both members of the resulting equation.

3. Collect terms and, if the equation still contains radical expressions, repeat (1) and (2) until the equation is free from radicals.

4. Solve the equation of (3).

5. Check each of the results by substitution in the original equation, rejecting all extraneous roots.

EXERCISES (127)

Solve and check, rejecting all extraneous roots:

1. $\sqrt{x + 5} = 21$ 2. $2\sqrt{x + 7} = 10$ 3. $\sqrt{3x - 7} - 20 = 0$

4. $2\sqrt{3y + 1} = \sqrt{8y + 20}$ 5. $\sqrt[3]{x^2 - 1} = 2$
6. $\sqrt{x^2 - 4} + 2 = x$ 7. $\sqrt{x + 2} = x - 4$
8. $6 - x = 3\sqrt{x - 2}$ 9. $\sqrt{y + 7} + y = 12$
10. $x - 4 = (x - 2)^{\frac{1}{2}}$ 11. $x - 4 = (4x - 11)^{\frac{1}{2}}$
12. $x^{\frac{1}{2}} + 7 = (x - 7)^{\frac{1}{2}}$ 13. $(2x + 5)^{\frac{1}{2}} - 5 = -(2x)^{\frac{1}{2}}$
14. $\sqrt[3]{x - 2} = \sqrt{x - 2}$ 15. $x^{\frac{1}{2}} - 5 = (x - 5)^{\frac{1}{2}}$

16. $\sqrt{x + 3} + \sqrt{x - 2} = \sqrt{2x + 1}$
17. $\sqrt{x + 4} + \sqrt{x - 4} = 4$
18. $\sqrt{x + 1} + \sqrt{x - 1} = \sqrt{14x - 8}$

128. Poor Elmer! "Hey, George!" said Elmer, "I've got your phony algebra figured out. You stuck in some extraneous roots. You can make anything equal anything that way!"

"Now, Elmer," said George, "you're making a big mistake. Here, I'll show you I can prove my statement with numbers, so you'll surely believe it. What does $\frac{1}{-1}$ equal?"

"It equals -1," said Elmer.

"And what does $\frac{-1}{1}$ equal?" asked George.

"It equals -1 also."

"Then $\frac{1}{-1} = \frac{-1}{1}$, doesn't it?"

"Yes," said Elmer, "it does."

"All right," said George, "take the square root of each side and you have $\frac{1}{\sqrt{-1}} = \frac{\sqrt{-1}}{1}$, or $\frac{1}{i} = \frac{i}{1}$. O.K.?"

"Yes," said Elmer, "that looks all right."

"Now," said George, "cross multiply, obtaining $1 = i^2$. Then, adding 1 to each side, you see that $2 = i^2 + 1$. But $i^2 = -1$; hence $2 = -1 + 1 = 0$; so your old two-dollar bill was worthless."

Poor Elmer! (A hint for Poor Elmer! If $x^2 = y^2$, does $x = y$? Suppose $x = 2$ and $y = -2$!)

REVIEW QUESTIONS

1. How may extraneous roots be introduced in solving an equation?
2. How may roots be lost during the solution of an equation?
3. Why must one check each root obtained in solving an irrational equation?
4. Can $\sqrt{x} + \sqrt{x-2} = 0$? Why or why not?

CHAPTER 20

PROGRESSIONS

Much of the mathematics of finance (simple interest, compound interest, annuities, taxes, investments, etc.) is related to the theory of *progressions.*

129. Arithmetic Progressions. An *arithmetic progression* is a sequence of numbers, called *terms,* each of which differs from the one before it by the same amount. For example, the numbers 10, 12, 14, 16, 18, 20, 22, . . . form an arithmetic progression; for each of them exceeds the preceding one by the amount 2, which is called the *common difference d.* An arithmetic progression is thus of the form $a, a + d, a + 2d, a + 3d, \cdots$

A practical example of an arithmetic progression is the sequence formed by the values of an investment at simple interest:

Example 1. If the amount $1,000 is invested at 3%, the yearly values of the investment, including both principal and interest, are $1,000, $1,030, $1,060, $1,090, etc., which form an arithmetic progression. It is assumed that the interest payments merely accumulate and are not reinvested. The common difference in this case is $30.

Example 2. Find the value of x for which the three quantities $2x + 1$, $x + 4$, $4x + 5$ form an arithmetic progression.

Solution: If these three quantities form an arithmetic progression,

then $$(2x + 1) - (x + 4) = (x + 4) - (4x + 5)$$
or $$2x + 1 - x - 4 = x + 4 - 4x - 5$$
Collecting, $$x - 3 = -3x - 1$$
or $$4x - 2 = 0 \quad \text{and} \quad x = \tfrac{1}{2}$$
Checking, $$2x + 1 = 2, \quad x + 4 = 4\tfrac{1}{2}, \quad 4x + 5 = 7$$
$$2 - 4\tfrac{1}{2} = -2\tfrac{1}{2} \quad \text{and} \quad 4\tfrac{1}{2} - 7 = -2\tfrac{1}{2}$$

130. Geometric Progressions. A geometric progression is a sequence of terms in which the ratio of each term to the one before it is the same. The *common ratio r,* the ratio of each term to the preceding one, is found by dividing any term by the one that precedes it. The terms 2, 6, 18, 54, 162, $\cdots$ form a geometric progression, as do also 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$ and 3, -6, 12, -24,

48, -96, $\cdots$. The common ratios for these three progressions are 3, $\frac{1}{2}$, and -2, respectively (check these values). In general, a geometric progression is of the form a, ar, ar^2, ar^3, ar^4, . . . , with r the common ratio.

A practical example of a geometric progression is the value of an investment at compound interest:

Example 1. The sum of \$1,000 is invested at 2%, compounded annually. The value of the investment after 1 year is \$1,000 $\times$ 1.02; after 2 years, \$1,000 $\times$ 1.02 $\times$ 1.02; after 3 years, \$1,000 $(1.02)^3$; and after n years, \$1,000$(1.02)^n$. Since the ratio of each of these terms to the preceding one is 1.02, they form a geometric progression.

Example 2. For what values of x do the terms $2x + 3$, x, $2x - 3$, form a geometric progression?

Solution: If the three terms form a geometric progression, then

$$\frac{x}{2x+3} = \frac{2x-3}{x}, \qquad \text{or} \qquad 4x^2 - 9 = x^2$$

Then

$$3x^2 = 9 \qquad \text{or} \qquad x^2 = 3$$

so that $x = \sqrt{3}$ and $x = -\sqrt{3}$ are the desired values. Checking, the three terms are $2\sqrt{3} + 3$, $\sqrt{3}$, $2\sqrt{3} - 3$

$$\frac{\sqrt{3}}{2\sqrt{3}+3} = \frac{2\sqrt{3}-3}{\sqrt{3}} \qquad \text{or} \qquad 12 - 9 \equiv 3 \qquad \text{checks}$$

EXERCISES (130)

Classify each of the following sequences as forming an arithmetic progression, a geometric progression, or neither, and write the next term in each arithmetic or geometric progression.

1. 4, 7, 10, 13, 16
2. 13, 9, 5, 1, -3
3. 2, -10, 50, -250
4. 96, -24, 6, $-1\frac{1}{2}$
5. 0, 1, 3, 9
6. 8, 4, 2, -2, -4
7. 1, $\frac{5}{2}$, 4, $\frac{11}{2}$
8. -3.9, -2.6, -1.3, 0
9. $\sqrt{3}$, 3, $3\sqrt{3}$, 9
10. 8, 12, 18, 27

Determine the values of x for which the following are arithmetic progressions:

11. 7, 4, x
12. 2, x, -12
13. 1, $x + 2$, $3x - 4$
14. $x + 3$, 5, $2x - 5$
15. $2x + 3$, $x + 6$, $4x + 7$
16. $5x - 1$, $3(x + \frac{1}{2})$, $3x - 2$
17. $x - 1$, $\frac{x}{16}$, $\frac{x+1}{-2}$
18. $4x + 23$, $2x + 7$, $x - 1$
19. $x^2 - 1$, $4 - 3x$, $x + 1$
20. $x^2 + 1$, $2x - 2$, $3x - 7$

Determine the values of x for which the following are geometric progressions:

21. $54, 36, x$
22. $2, x, 8$
23. $x + 3, x, x - 6$
24. $2x^2 + 3x + 1, x - 1, \frac{1}{2}$
25. $x - 2, x - 5, x - 7$
26. $x - 3, x + 2, x + 5$
27. $4x - 1, 2x + 1, x - 1$
28. $x - 1, x + 2, x + 2$
29. $x + 4, 2x, x - 1$
30. $x + 1, 2x + 2, 5x + 2$

131. Nth Term of an Arithmetic Progression. If the first term of an arithmetic progression is a and the common difference is d, the terms of the progression are

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \text{ etc.}$$

Note that the coefficient of d is always one less than the number n of the term. For example, the coefficient of d in the fourth term is 3, or $4 - 1$. If the nth term is represented by l, then

$$\boxed{l = a + (n - 1)\,d}$$

For example, the fiftieth term in the progression $a, a + d, a + 2d$, etc., is $l = a + 49d$. If any three of the numbers l, a, n, d, are known for a given progression, the remaining one can be determined.

Example 1. Find the twenty-third term of the progression $7, \frac{17}{2}, 10, \frac{23}{2}$, etc.

Solution: This sequence is an arithmetic progression in which $a = 7$ and $d = \frac{3}{2}$. In order to find the twenty-third term $n = 23$, we write

$$l = a + (n - 1)d \qquad \text{or} \qquad l = 7 + 22(\tfrac{3}{2}) = 7 + 33 = 40$$

Example 2. If the third and seventh terms of an arithmetic progression are 5 and -31, respectively, what are the first four terms?

Solution: With 5 as the nth term, $n = 3$ and $l = 5$.
Then $l = a + (n - 1)d$ becomes

$$5 = a + 2d \tag{1}$$

Using -31 as the nth term ($n = 7$),

$$-31 = a + 6d \tag{2}$$

Solving (1) and (2) for a and d gives $d = -9$ and $a = 23$. The first four terms of the progression, therefore, are

$$23, 14, 5, -4 \qquad \text{(check these)}$$

Example 3. Is 79 a term of the progression 7, $8\frac{1}{3}$, $9\frac{2}{3}$, 11, etc.?

Solution: If 79 is the nth term, then

$$79 = a + (n - 1)d = 7 + (n - 1)(\tfrac{4}{3}), \text{ since } d = \tfrac{4}{3}$$

Then $79 = 7 + \frac{4}{3}n - \frac{4}{3}$, or $\frac{4}{3}n = 73\frac{1}{3}$, and $n = 55$. Thus 79 is the fifty-fifth term.

Example 4. Is 38 a term of the progression of Example 3?

Solution: $38 = 7 + (n - 1)\frac{4}{3}$ or $\frac{4}{3}n = 38 - 7 + \frac{4}{3}$

Then $$n = \tfrac{3}{4}(32\tfrac{1}{3}) = 24\tfrac{1}{4}$$

Since n must be an integer for each term, 38 is not a term of the progression.

EXERCISES (131)

In each progression, find the term indicated:

1. 4, 7, 10, . . . , 13th term
2. 6, 8, 10, . . . , 23d term
3. 55, 51, 47, . . . , 17th term
4. 154, 148, 142, . . . , 11th term
5. 2, $3\frac{1}{2}$, 5, . . . , 27th term
6. $5\frac{1}{2}$, $4\frac{3}{4}$, 4, . . . , 15th term
7. $-\frac{9}{2}$, -2, $+\frac{1}{2}$, · · · , 19th term
8. 3.7, 4.1, 4.5, . . . , 32d term
9. 7.8, 7.5, 7.2, . . . , 54th term
10. 3.027, 3.098, 3.169, . . . , 47th term

132. Arithmetic Means. In an arithmetic progression, all the terms between the first term and the last term are called *arithmetic means*. For example, in the progression -5, -3, -1, $+1$, $+3$, the terms -3, -1, $+1$ are the three arithmetic means between -5 and $+3$. Likewise, in the arithmetic progression 2, 5, 8, 11, the numbers 5 and 8 are the two arithmetic means between 2 and 11.

Example 1. Find five arithmetic means between -9 and 54.

Solution: The desired arithmetic progression contains seven terms, since there are to be five arithmetic means plus the first and last terms.

Then	$l = a + (n - 1)d$
becomes	$54 = -9 + (7 - 1)d$
from which	$d = \frac{63}{6} = 10.5$

The desired arithmetic progression is -9, 1.5, 12, 22.5, 33, 43.5, 54, the five arithmetic means being 1.5, 12, 22.5, 33, and 43.5.

The most significant case is that of *one* arithmetic mean between two numbers, which is the *arithmetic average* of the two numbers.

If M represents the arithmetic mean between a and b, so that the sequence a, M, b is an arithmetic progression, then

$$M - a = b - M \qquad \text{or} \qquad M = \frac{a+b}{2}$$

Example 2. Find the arithmetic mean of (between) 17 and 46.

Solution: $$M = \frac{17+46}{2} = \frac{63}{2} = 31.5$$

EXERCISES (132)

Between each pair of numbers, insert arithmetic means as indicated:

1. 19 and 35, 4 means
2. 9 and -31, 7 means
3. 6 and 7, 2 means
4. 9 and 11, 3 means
5. 8.4 and 10.8, 11 means
6. -2.4 and 6, 5 means
7. 24 and 31, one mean
8. 16.5 and 34, one mean
9. -25 and 43, 3 means
10. 82 and -3, one mean

133. Sum of N Terms of an Arithmetic Progression. If S_n represents the sum of n terms of an arithmetic progression, then

$$S_n = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l$$

Observe the last three terms, noting that the next-to-last term is d less than l, the second from last is $2d$ less than l, etc. Let us write the same progression twice, reversing its order the second time,

$$S_n = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l$$
$$S_n = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a$$

If we add term by term, the result is

$$2S_n = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l)$$

or $2S_n = n(a+l)$, since there are n terms. Dividing by 2,

$$\boxed{S_n = \frac{n}{2}(a+l)}$$

Or, in words, the sum of n terms of an arithmetic progression equals n times the *arithmetic average* of the first and last terms. The use of this formula will be illustrated by examples:

Example 1. Find the sum of the first 82 terms of the progression

$$-17, -14, -11, -8, \text{ etc.}$$

Solution: This sequence is an arithmetic progression for which $a = -17$ and $d = 3$.

Then $\qquad S_n = \frac{n}{2}(a + l) = \frac{82}{2}(-17 + l)$

But $\qquad l = a + (n - 1)d = -17 + (81 \times 3) = 226$

hence $\qquad S_n = \frac{82}{2}(-17 + 226) = 41 \times 209 = 8{,}569$

(Note that to obtain this result by adding 82 terms would be a long process.)

Example 2. Find the common difference and the number of terms in an arithmetic progression whose first and last terms are -8 and 100, respectively, and whose sum is 1,702.

Solution: Since $\qquad S_n = \frac{n}{2}(a + l)$

we can write $1{,}702 = \frac{n}{2}(-8 + 100) \qquad$ or $\qquad n = \frac{3{,}404}{92} = 37$

Then, since $\qquad l = a + (n - 1)d$

$$100 = -8 + 36d \qquad \text{and} \qquad d = \tfrac{108}{36} = 3$$

Example 3. How many terms of the progression $-7, -5, -3$, etc., must be taken in order to equal or exceed 100?

Solution: $S_n = \frac{n}{2}(a + l) \qquad$ or $\qquad 100 = \frac{n}{2}(-7 + l) \qquad (1)$

Also $\qquad l = a + (n - 1)d \qquad$ or $\qquad l = -7 + (n - 1)2 \qquad (2)$

Equation (1) can be written as $l = \dfrac{200}{n} + 7$

Equation (2) can be written as $l = 2n - 9$

Then $\qquad 2n - 9 = \dfrac{200}{n} + 7 \qquad$ or $\qquad 2n^2 - 16n - 200 = 0$

$$D(2), \qquad n^2 - 8n - 100 = 0$$

Then $n = \dfrac{8 \pm \sqrt{64 + 400}}{2} = 4 \pm \sqrt{116} \doteq 4 \pm 10.77 = 14.77$ or -6.77

Since $n = -6.77$ has no meaning, we take $n = 14.77$ as the solution. If the value of n is not an integer, the nearest (larger) integer represents the number of terms required to exceed the specified sum, since in that case no number of terms produces exactly the desired sum.

EXERCISES (133)

In each progression, find the sum of n terms as indicated:

1. 15 terms of 3, 7, 11, . . .
2. 31 terms of -8, -3, 2, . . .
3. 7 terms of 27, 24, 21, . . .
4. 6 terms of 19, 11, 3, . . .
5. 26 terms of 14, $12\frac{1}{2}$, 11, . . .
6. 17 terms of 4, $4\frac{1}{3}$, $4\frac{2}{3}$, . . .
7. 12 terms of 1.7, 4.2, 6.7, . . .
8. 19 terms of 5.7, 4.3, 2.9, . . .
9. 7 terms of 2.72, 2.58, 2.44, . . .
10. 40 terms of 2.72, 2.58, 2.44, . . .

11. In an arithmetic progression, $d = 3$ and the thirteenth term is 44. Find the fifth term.

12. In an arithmetic progression, $d = 2.5$ and the eighth term is 32. Find the third term.

13. The fifth term of an arithmetic progression is 24, and the twelfth term is 52. Find a and d.

14. The seventh term of an arithmetic progression is 11, and the twelfth term is -14. Find a and d.

15. If $a = 7$, $d = 3\frac{1}{2}$, $l = 42$, find n and S.

16. If $d = 2$, $l = 45$, $n = 17$, find a and S.

17. If $a = -5$, $d = 1\frac{1}{2}$, $n = 8$, find l and S.

18. If $a = 5$, $l = 19$, $S = 84$, find d and n.

19. If $a = 17$, $n = 5$, $S = 72$, find d and l.

20. If $d = 4$, $n = 8$, $S = 120$, find a and l.

134. Nth Term of a Geometric Progression. A geometric progression can be represented by the sequence

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ \text{etc.},$$

in which it can be seen that the exponent of r is always one less than n, the number of the term. Thus, the fourth term is ar^3, the seventh term is ar^6, and the nth term l is

$$\boxed{l = ar^{n-1}}$$

If three of the four numbers l, a, r, n, are known, the other one can be determined.

Example 1. Find the seventh term of the progression

$$27,\ 18,\ 12,\ \ldots.$$

Solution: This is a geometric progression in which $a = 27$, $r = \frac{2}{3}$. For $n = 7$, therefore, $l = ar^{n-1}$ becomes

$$l = 27\left(\frac{2}{3}\right)^6 = 3^3\left(\frac{2^6}{3^6}\right) = \frac{2^6}{3^3} = \frac{64}{27}$$

Example 2. The first term of a geometric progression is 2.75, and the fifth term is 75. Find the third term.

Solution: Substituting $a = 2.75$, $l = 75$, $n = 5$ in $l = ar^{n-1}$,

$$75 = 2.75r^4 \quad \text{or} \quad r^4 = \frac{75}{2.75}$$

This equation cannot be solved by inspection; hence logarithms should be used.

$$\log r = \frac{1}{4} \log \left(\frac{75}{2.75}\right)$$

$$\begin{aligned} \log 75 &\doteq 1.8751 \\ \log 2.75 &\doteq 0.4393 \\ \hline \log\left(\frac{75}{2.75}\right) &\doteq 1.4358 \end{aligned}$$

$$\log r \doteq \frac{1.4358}{4} \doteq 0.3590$$

$$r \doteq \text{antilog } 0.3590 \doteq 2.286$$

The third term is obtained by substituting $n = 3$ in

$$\begin{aligned} l &= ar^{n-1} = 2.75r^2 \\ \log l &= \log 2.75 + 2 \log r \\ \log 2.75 &\doteq 0.4393 \\ 2 \log r \doteq 2(0.3590) &= 0.7180 \\ \hline \log l &\doteq 1.1573 \\ l &\doteq 14.37, \text{ which is the third term} \end{aligned}$$

EXERCISES (134)

In each progression, find the term indicated:

1. 2, 4, 8, . . . , 6th term
2. $\frac{1}{6}, \frac{1}{2}, \frac{3}{2}$, . . . , 7th term
3. $\frac{1}{4}, -\frac{1}{2}$, 1, . . . , 9th term
4. $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}$, . . . , 11th term
5. $\frac{1}{100}, \frac{1}{10}$, 1, . . . , 9th term
6. 2.064, 1.032, .516, . . . , 9th term
7. 2, 2.22, 2.4642, . . . , 8th term
8. 3.03, 3, $\frac{3}{1.01}$, . . . , 7th term

9. The first term of a geometric progression is 2, and the fifth term is 162. Find the common ratio.

10. In a geometric progression, the first term is .2, and the third term is 5. Find the sixth term.

11. The third and fourth terms of a geometric progression are 2 and 5, respectively. Write the first two terms.

12. The first term of a geometric progression is $\frac{1}{6}$, the third term is $\frac{2}{3}$, and the last term is $-\frac{64}{3}$. Find the fourth term.

135. Geometric Means. In a geometric progression, all the terms between the first and last terms are called *geometric* means between those two terms. For example, in the progression $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, the numbers $\frac{1}{4}$, $\frac{1}{2}$, 1 are the three geometric means between $\frac{1}{8}$ and 2. In the progression 5, 15, 45, 135, the numbers 15 and 45 are the two geometric means between 5 and 135.

Example 1. Find the two geometric means between 2 and 250.

Solution: The two geometric means, together with the first and last terms, will constitute a geometric progression of four terms. Hence we write

$$l = ar^{n-1} \qquad \text{as} \qquad 250 = 2r^{4-1} = 2r^3$$

Then $$r^3 = \frac{250}{2} = 125 \qquad \text{and} \qquad r = \sqrt[3]{125} = 5$$

The geometric progression is 2, 10, 50, 250, with 10 and 50 the desired geometric means.

The most significant case is that of *one* geometric mean between two numbers. If M is the geometric mean between a and b, then $\frac{M}{a} = r$, and $\frac{b}{M} = r$, or $\frac{M}{a} = \frac{b}{M}$. Then $M^2 = ab$, or $M = \pm\sqrt{ab}$.* Since $\frac{M}{a} = \frac{b}{M}$ is equivalent to $\frac{a}{M} = \frac{M}{b}$, it can be seen that the single geometric mean is identical with the *mean proportional.*

EXERCISES (135)

1. Find a positive geometric mean between 2 and 18.
2. Find a geometric mean between $2x$ and $32x^5$.
3. Find a mean proportional between 7 and 8.
4. Insert two geometric means between 3 and 24.
5. Insert two geometric means between 5 and 90.
6. Insert three geometric means between 9 and $\frac{1}{9}$.
7. Insert three geometric means between 3 and 70.
8. The arithmetic mean between two numbers is 130, and their geometric mean is 5. Find the numbers.

136. Sum of N Terms of a Geometric Progression. If S_n represents the sum of n terms of a geometric progression, then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1} \qquad (1)$$

(Examine the exponents in the last three terms.) Multiplying both sides of the equation by r,

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \qquad (2)$$

If we now subtract this equation, term by term, from the first equation, the result is

$$S_n - rS_n = a - ar^n \quad \text{(check this subtraction)}$$

Then $S_n(1 - r) = a - ar^n$, or

$$\boxed{S_n = \frac{a - ar^n}{1 - r}}$$

Also, since $\quad l = ar^{n-1}, \quad rl = ar^n$

Substituting rl for ar^n,

$$\boxed{S_n = \frac{a - rl}{1 - r}}$$

The use of these equations will now be illustrated.

Example 1. Find the sum of 2, 6, 18, . . . , to 6 terms.

Solution: $S_n = \dfrac{a - ar^n}{1 - r} = \dfrac{2 - 2(3^6)}{1 - 3} = \dfrac{2 - 2(3^6)}{-2} = 3^6 - 1$

But $3^6 = 729$; hence $S_n = 729 - 1 = 728$.

Example 2. If $a = 54$, $l = \frac{2}{3}$, $S_n = 80\frac{2}{3}$; find r and n.

Solution: $\quad S_n = \dfrac{a - rl}{1 - r} \quad$ or $\quad 80\frac{2}{3} = \dfrac{54 - r(\frac{2}{3})}{1 - r}$

From this, $\quad \frac{242}{3}(1 - r) = 54 - \frac{2}{3}r$

M(3), $\quad 242(1 - r) = 162 - 2r$

$$240r = 242 - 162 = 80$$

or $\quad r = \frac{1}{3}$

Substituting in $\quad l = ar^{n-1}$

(in order to find n),

$$\frac{2}{3} = 54(\tfrac{1}{3})^{n-1} \quad \text{or} \quad (\tfrac{1}{3})^{n-1} = \tfrac{1}{81}$$

But $\frac{1}{81} = (\frac{1}{3})^4$; hence

$$n - 1 = 4 \quad \text{or} \quad n = 5 \text{ terms}$$

EXERCISES (136)

In each progression, find the sum of n terms as indicated:

1. $\frac{1}{2}, 1, 2, \ldots$, 10 terms
2. $\frac{1}{9}, \frac{1}{3}, 1, \ldots$, 7 terms
3. 12, 6, 3 . . . , 9 terms
4. 2, 5, 12.5, . . . , 8 terms
5. $\sqrt{2}, 2, 2\sqrt{2}, \ldots$, 8 terms
6. 500, 50, 5, . . . , 7 terms
7. 2.064, 1.032, .516, . . . , 7 terms
8. 2, 2.22, 2.4642, . . . , 6 terms
9. If $S_n = 171$, $l = 256$, $r = -2$, find n and a.
10. If $S_n = 186$, $r = 2$, $a = 6$, find l and n.
11. If $S_n = 2\frac{5}{8}$, $r = \frac{1}{2}$, $a = \frac{4}{3}$, find l and n.
12. If $S_n = 508$, $r = 2$, $a = 4$, find l and n.

137. Sum of an Infinite Geometric Progression.

Example 1. Suppose that a rubber ball is dropped from a height of 75 ft. above the ground and that each time it strikes the ground it bounces to $\frac{2}{5}$ the height from which it fell. How far does the ball travel after it first strikes the ground, assuming that it keeps bouncing until its motion is imperceptible?

The distances traveled by the ball on successive bounces form the progression 60, 24, $\frac{48}{5}$, $\frac{96}{25}$, . . . (check). The total distance represented by n terms is

$$S_n = 60 + 24 + \tfrac{48}{5} + \tfrac{96}{25} + \cdots \text{ to } n \text{ terms}$$

or

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

But this relation was proved equivalent to

$$S_n = \frac{a - rl}{1 - r}$$

In this example, $a = 60$ and $r = \frac{2}{5}$.

Then

$$S_n = \frac{60 - \frac{2}{5}l}{1 - \frac{2}{5}} = \frac{300 - 2l}{3} = 100 - \tfrac{2}{3}l$$

This means that, after n bounces, the ball has traveled a distance $100 - \frac{2}{3}l$. This distance is always less than 100 ft., less by just two-thirds of the distance traveled on the preceding bounce. For example, the ball travels 60 ft. on the first bounce and 24 ft. on the second bounce, or a total of $60 + 24 = 84$ ft. for the first two bounces. Note that $84 = 100 - 16$, or $100 - \frac{2}{3}(24)$, which is $100 - \frac{2}{3}l$.

Examine the implications of the relation $S_n = 100 - \frac{2}{3}l$. (1) S_n can never be more than 100 ft. (2) Since l will eventually become imperceptibly small, S_n will reach 100 ft., *for all practical purposes*. For example, on the tenth bounce,

$$l = ar^{n-1} = 60\left(\tfrac{2}{5}\right)^9 \doteq .0157 \text{ ft. (check by logarithms).}$$

On the fifteenth bounce,

$$l = 60\left(\tfrac{2}{5}\right)^{14} \doteq .0000161 \text{ ft.}$$

and on the thirtieth bounce,

$$l = 60\left(\tfrac{2}{5}\right)^{29} \doteq .000000000172 \text{ ft.,}$$

Note that S_{30} is (100 − .000000000172) ft. or 99.999999999828 ft. Since S_n cannot exceed 100 ft., and since the amount it lacks being 100 ft. becomes smaller and smaller, we say that it is approaching 100 ft. *as a limit.*

Now consider the relation

$$S^n = \frac{a - ar^n}{1 - r}$$

This can be written as $S_n = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}$

For a given value of a, and for r numerically less than 1, the term $\dfrac{ar^n}{1 - r}$ decreases as the number of terms (n) increases, since r^n decreases as n becomes large. For example, for $r = \frac{1}{2}$ and $n = 30$,

$$\left(\frac{1}{2}\right)^{30} = \frac{1}{1,073,741,824}$$

As n increases, therefore, the term $\dfrac{ar^n}{1 - r}$ approaches *zero* as a limit; therefore S_n approaches the definite value $\dfrac{a}{1 - r}$. We indicate this by writing $S_n \underset{n\to\infty}{\to} \dfrac{a}{1 - r}$, meaning, "$S_n$ approaches $\dfrac{a}{1 - r}$ as n approaches *infinity*." The same information can be indicated by

$$S_\infty = \frac{a}{1 - r}$$

where S_∞ is read, "The sum to infinity . . . ." Use this formula to check the value of S_∞ for the case of the bouncing ball. It is

important to realize that S_∞ has no meaning unless r is numerically less than 1.

Example 2. Find the sum, to infinity, of the terms of the progression $5.5, 5, \frac{5}{1.1}, \ldots$.

Solution: $a = 5.5$ and $r = \frac{1}{1.1}$

Then $$S_\infty = \frac{5.5}{1 - \frac{1}{1.1}} = \frac{5.5 \times 1.1}{1.1 - 1} = \frac{6.05}{.1} = 60.5$$

When one changes a fraction to a decimal, the result either divides without a remainder after a certain number of decimal places, or else the figures repeat in groups, forming what is called a *repeating decimal.* Thus, $\frac{7}{8} = .875000 \cdots$, but

$$\frac{1.6}{11} = 1.454545454545 \cdots$$

and $$\tfrac{34}{7} = 4.857142857142857142857142 \cdots$$

Example 3. Find the value of the repeating decimal .63636363 . . . .

Solution: This repeating decimal is equivalent to the sum of the geometric progression: $\frac{63}{100}, \frac{63}{10{,}000}, \frac{63}{1{,}000{,}000}$, etc., in which

$$a = .63 \quad \text{and} \quad r = .01$$

The "sum to infinity" of the progression is

$$S_\infty = \frac{a}{1 - r} = \frac{.63}{1 - .01} = \frac{.63}{.99} = \frac{7}{11}$$

This shows that the repeating decimal .636363 . . . is a rational number (*i.e.*, expressible in terms of integers), even though its exact value cannot be expressed as a decimal fraction.

EXERCISES (137)

Find the sum, to infinity, of each of the following:

1. $1, \frac{1}{2}, \frac{1}{4}, \ldots$
2. $1, \frac{1}{10}, \frac{1}{100}, \ldots$
3. $10, 2, \frac{2}{5}, \ldots$
4. $3, 1, \frac{1}{3}, \ldots$
5. $x, \frac{x}{3}, \frac{x}{9}, \ldots$
6. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \cdots$
7. $-3, -\frac{1}{2}, -\frac{1}{12}, \cdots$
8. $\frac{1}{3}, -\frac{1}{5}, \frac{3}{25}, \cdots$
9. $3, -\frac{3}{4}, \frac{3}{16}, \cdots$
10. $6, -2, \frac{2}{3}, \cdots$

Evaluate the following repeating decimals:

11. .66666 . . .	12. .5555 . . .	13. .1111 . . .
14. .7777 . . .	15. .522222 . . .	16. 2.8181 . . .
17. 1.324545 . . .	18. 3.56969 . . .	19. 2.3636 . . .
20. 5.408111 . . .	21. .141141 . . .	22. 2.381381 . . .

138. Stated Problems Involving Progressions. In the following problems, first classify the sequence involved (as an arithmetic progression or a geometric progression) then apply the methods of this chapter.

1. A dealer sold a radio by means of a raffle in which 100 tickets, numbered 1 to 100, were taken from a "punchboard." If the price of each ticket (in cents) corresponded to its number, and if the radio and punchboard cost the dealer $35, determine his profit.

2. In Prob. 1, if the dealer had distributed 150 tickets (numbered 1 to 150), and if the even tickets were free (the odd ones being bought on the same basis as in Prob. 1), what would have been the dealer's profit?

3. In order to provide a fund for his son's college education, a man deposits $100 each year, the deposits bearing 4% simple interest. What is the total value of the investment after the eighteenth deposit has been made?

4. In Prob. 3, what would have been the result if the interest had been compounded annually, *i.e.*, reinvested each year?

5. If a ball rolling down an inclined plane rolls 1 ft. during the first second, 3 ft. during the next second, 5 ft. during the next, etc., how far will it roll in 10 sec.?

6. How far does the ball of Prob. 5 roll during the tenth second? during the fifth?

7. A rubber ball is dropped from a height of 45 ft., and each time it strikes the ground it bounces to $\frac{2}{3}$ the height from which it fell. How far does it travel in all? (In order to avoid confusion, first determine how far the ball travels after first striking the ground; then add 45 ft. This is necessary because 45 is not one of the terms in the progression, which starts with $a = 30$.)

8. In Prob. 7, how far does the ball travel on the fifth bounce?

9. A freely falling object drops 16 ft. the first second, 48 ft. the next second, 80 ft. the next, etc. If a bomb dropped from an airplane reaches the ground in 12 sec., how high is the plane, the resistance of the air being neglected?

10. How long would be required for a bomb to fall to the earth from a height of 5,184 ft.?

11. A rock is dropped into an abandoned mine shaft, and the sound of its impact at the bottom is heard 4.7 sec. later. If the time required for the sound to travel is negligible, how deep is the shaft? NOTE: Although n is not an integer in this case, the same method may be used.

12. As she placed her husband's birthday cake on the table, the mother made the familiar remark, "I wish I had a dime for every candle I have put on a birthday cake." She had reared four children, now aged 11, 14, 17, and 19 years, respectively, and had always made a birthday cake for each of them (with each year represented by a candle on the cake). Also, she had started the same custom for her husband at his twenty-third birthday and had kept it up until the present, his forty-seventh birthday. How much would her "dimes" be worth?

13. A man is fencing a rectangular plot 400 by 100 ft. He spaces the posts exactly 6 ft. apart (along the perimeter), except at two opposite corners, where the "slack" is taken up. How many posts will he need?

14. A clock strikes a two-note chime once at one o'clock, twice at two o'clock, etc., and a single chime once at quarter past, twice at half past, and three times at quarter of. How many notes does it strike in 12 hr.?

15. Find the approximate value of \$5 invested for 100 years at 6% interest, compounded annually. Use logarithms.

16. How long will it take x dollars to double itself at 4% simple interest? To triple itself? (The answer is not an integer.)

17. How long will it take x dollars to double itself at 4% interest, compounded annually? To triple itself? (Note: *Fractions* of a year must be computed at *simple* interest.)

18. As an automobile is moving along a level stretch of pavement, the driver disengages the clutch, allowing the automobile to coast. If in the first second the automobile travels 80 ft., and if in each succeeding second it travels 20% less than in the preceding second, how far will the automobile coast?

19. At each stroke, a vacuum pump removes 3% of the (remaining) air in a container. What fraction of the air will remain after 100 strokes?

20. One person contributes a dime to a certain charity, then sends letters to three friends, asking them to contribute a dime each and write letters to three friends, etc. If the "chain" holds until nine sets of letters have been written, how much money will the charity receive? Assume that those who *receive* letters in the ninth set do not respond.

21. A 1-gal. can of alcohol is emptied of one-third of its contents and refilled with water. This is done ten times. What is the resulting alcoholic content, in per cent (by volume)?

22. Suppose you have your choice of

(*a*) 1 cent for the first month of your life, 2 cents for the second month, 3 cents for the next, etc., or

(*b*) \$1 for the first year, \$2 for the next, \$3 for the next, etc., or

(*c*) 1 cent for the first year of your life, 2 cents for the second year, 4 cents for the third, etc.

Which do you choose? In each case, use your age at your last birthday.

REVIEW QUESTIONS

1. Define an arithmetic progression.
2. Define a geometric progression.
3. Prove that, in a geometric series,

$$S_n = \frac{a - ar^n}{1 - r} \quad \text{is equivalent to} \quad S_n = \frac{a - rl}{1 - r}$$

4. Prove that the sum of the integers 1 to n is $\dfrac{n(n + 1)}{2}$.

CHAPTER 21

BINOMIAL THEOREM, APPROXIMATIONS

139. Powers of Binomials. The expanded forms of various integral powers of the binomial expression $(a + b)$ are shown below. They can be obtained by successive multiplications.

$$
\begin{aligned}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 - b^5
\end{aligned}
$$

The expansions of $(a + b)^n$, where $n = 1, 2, 3, 4, 5$, as in the above cases, can be written (without successive multiplications) by application of the following set of rules:

In the expansion of $(a + b)^n$, with n an integer,

1. The first term is a^n (check this in the examples above).
2. The coefficient of the second term is n.
3. The exponent of a decreases by 1 in each successive term, and the exponent of b increases by 1 (check).
4. If the numerical coefficient of any term is multiplied by the exponent of a in that term and then divided by one more than the exponent of b, the result is the coefficient of the *next* term. (Check this.)

This set of rules forms what is called the *binomial theorem.* It is valid for all positive integral values of n. With certain limitations, it is valid also for negative and fractional values of n.

Example 1. Expand $(a + b)^6$.

Solution: Applying the binomial theorem, we observe that the first term is a^6. Note that the exponent of b in the first term is zero, *i.e.*, $a^6 = a^6b^0$. The coefficient of the second term is 6, the exponent of a in the second term is $6 - 1 = 5$, and the exponent of b is $0 + 1 = 1$. Thus the second term is $6a^5b$. The coefficient of the third term is $\frac{6 \times 5}{2} = 15$ (note how the number 2 is obtained), and the complete

third term is $15a^4b^2$. The fourth term is $20a^3b^3\left(\text{note that } \frac{15 \times 4}{3} = 20\right)$. Applying this method successively, we obtain

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Check this example, noting the symmetry of the coefficients (with respect to the middle term). Observe that the coefficients of the next term after b^6 is $\frac{(6)(0)}{7} = 0$; therefore the expansion stops at that term.

EXERCISES (139)

Expand by means of the binomial theorem:

1. $(a + b)^6$ 2. $(a + b)^7$ 3. $(x + y)^4$
4. $(x + y)^5$ 5. $(c + d)^8$ 6. $(c + d)^9$

Write only the first four terms of the following:

7. $(a + b)^{15}$ 8. $(a + b)^{21}$ 9. $(x + y)^{17}$
10. $(r + s)^{11}$ 11. $(c + d)^{20}$ 12. $(c + d)^{40}$

140. More Specific Cases. The procedure involved in expanding binomials of the general type $(a + b)^n$ will now be applied to specific cases:

Example 1. Expand $(x - y)^4$.

Solution: In order to avoid confusion in connection with signs, retain $(-y)$ as a single number so that only positive terms will appear in the initial expansion; *viz.*, $(x - y)^4 = [x + (-y)]^4$, so that

$$(x - y)^4 = x^4 + 4x^3(-y) + 6x^2(-y)^2 + 4x(-y)^3 + (-y)^4$$

Now the final form of the expansion can be written

$$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

When arithmetic numbers appear as members of the binomial, write out their powers in the expansion:

Example 2. Expand $(1 - x)^5$.

Solution:

$$(1 - x)^5 = 1^5 + 5(1)^4(-x) + 10(1)^3(-x)^2 + 10(1)^2(-x)^3 + 5(1)(-x)^4 + (-x)^5$$

or $$(1 - x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$$

The method can be used also for evaluating powers of numbers.

Example 3. Expand $(1.02)^4$ by the binomial theorem.

Solution:

$$\begin{aligned}(1 + .02)^4 &= 1^4 + 4(1)^3(.02) + 6(1)^2(.02)^2 + 4(1)(.02)^3 + (.02)^4\\ &= 1 + .08 + 2.4 \times 10^{-3} + 3.2 \times 10^{-5} + 1.6 \times 10^{-7}\end{aligned}$$

Adding these terms, we have

$$\begin{array}{r} 1.00 \\ .08 \\ .0024 \\ .000032 \\ .00000016 \\ \hline (1.02)^4 = 1.08243216 \end{array}$$

The importance of this method lies in the fact that a very accurate approximation can often be obtained by use of only the first few terms. In this case, 1.0824 would be obtained by using only the first three terms.

Example 4. Expand $(2x - y)^4$.

Solution:

$$(2x - y)^4 = (2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$$

or
$$(2x - y)^4 = 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$$

Example 5. Expand $(2x - 3y)^5$.

Solution: Again, treat $2x$ and $(-3y)$ as single numbers:

$$\begin{aligned}(2x - 3y)^5 = (2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3\\ + 5(2x)(-3y)^4 + (-3y)^5\end{aligned}$$

or $(2x - 3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

When powers of numbers appear inside the binomial, it is best to substitute another symbol in the initial expansion:

Example 6. Expand $(x^2 + y)^4$.

Solution: In order to simplify the expansion, first replace x^2 by a, writing

$$(a + y)^4 = a^4 + 4a^3y + 6a^2y^2 + 4ay^3 + y^4$$

Then replace a by x^2, obtaining

$$(x^2 + y)^4 = x^8 + 4x^6y + 6x^4y^2 + 4x^2y^3 + y^4$$

The reason for using this procedure is that the coefficients depend on the power of (x^2), not on the power of x itself. One is likely to

forget this if he writes the initial expansion in terms of x^2, instead of replacing it by a.

EXERCISES (140)

Expand by the binomial theorem:

1. $(a - b)^5$	2. $(x - y)^4$	3. $(a - 2)^6$	4. $(4 - y)^3$
5. $(x + 1)^7$	6. $(1 - y)^6$	7. $(2a - b)^4$	8. $(3x - 2)^3$
9. $(1 - 2y)^5$	10. $(3x - 1)^4$	11. $(1.002)^3$	12. $(1.005)^4$

13. $(.997)^3$ HINT: Note that $.997 = 1 - .003$.

14. $(.9995)^4$	15. $(x^2 + y)^3$	16. $(x^3 - y)^4$	17. $(a^5 + b)^7$
18. $(2x^2 - y)^5$	19. $\left(\frac{x}{y} + \frac{y}{x}\right)^4$	20. $\left(\frac{a}{b} - \frac{b}{a}\right)^3$	

Evaluate to four decimal places:

21. $(1.002)^9$	22. $(1.0015)^8$	23. $(.992)^{12}$	24. $(.9975)^{15}$

141. Fractional and Negative Powers of Binomials. For positive integral values of the exponent n, the expansion of $(a + b)^n$ always contains $n + 1$ terms. For fractional or negative values of n, this is not the case.

Example 1. Expand $\frac{1}{(a + b)^2}$ by the binomial theorem.

Solution:
$$\frac{1}{(a + b)^2} = (a + b)^{-2}$$

By the binomial theorem,

$$\begin{aligned}(a + b)^{-2} &= a^{-2} - 2a^{-3}b + 3a^{-4}b^2 - 4a^{-5}b^3 + 5a^{-6}b^4 + \cdots \\ &= \frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} + \cdots\end{aligned}$$

and it can be seen that there is no place at which to stop, since the exponent of a will never reach zero and thus will never stop the expansion by producing a zero coefficient. For this reason, expansions of negative and fractional powers of binomials are referred to as expansions *in series*, since they always lead to a never-ending series of terms.

Example 2. Expand $\sqrt{a + b}$.

Solution:

$$\sqrt{a + b} = (a + b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}a^{-\frac{5}{2}}b^2 + \cdots$$

Check these examples in detail to make sure that the handling of the exponents is clearly understood.

Example 3. Expand $\dfrac{1}{\sqrt[3]{x+y}}$.

Solution:
$$\frac{1}{\sqrt[3]{x+y}} = (x+y)^{-\frac{1}{3}}$$

Expanding,

$$(x+y)^{-\frac{1}{3}} = x^{-\frac{1}{3}} - \tfrac{1}{3}x^{-\frac{4}{3}}y + \tfrac{2}{9}x^{-\frac{7}{3}}y^2 - \tfrac{14}{81}x^{-\frac{10}{3}}y^3 + \cdots$$

EXERCISES (141)

Expand in series, obtaining the first four terms:

1. $\dfrac{1}{x+y}$
2. $\dfrac{1}{(a+b)^3}$
3. $\dfrac{1}{(1+x)^5}$
4. $\dfrac{1}{(2+y)^4}$
5. $\dfrac{1}{\sqrt{a+b}}$
6. $\dfrac{1}{\sqrt[3]{x-y}}$
7. $\sqrt[3]{(a+b)^2}$
8. $\dfrac{1}{\sqrt[3]{(a+b)^2}}$
9. $\dfrac{1}{(x-y)^7}$
10. $(x-y)^{-\frac{3}{8}}$
11. $\sqrt{1+x}$
12. $\dfrac{1}{\sqrt{1+x}}$
13. $\dfrac{1}{1.002}$
14. $\dfrac{1}{(1.002)^2}$
15. $\sqrt{1.008}$
16. $\sqrt{.997}$

Obtain only three terms in the following:

17. $\sqrt[5]{1.0015}$
18. $\sqrt[7]{1.005}$
19. $\dfrac{1}{\sqrt[5]{1.02}}$
20. $\dfrac{1}{\sqrt[3]{1.006}}$

142. The Binomial Formula. The result of the expansion of $(a+b)^n$ by the binomial theorem is called the *binomial formula:*

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{1 \times 2} + \frac{n(n-1)(n-2)a^{n-3}b^3}{1 \times 2 \times 3} + \frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{1 \times 2 \times 3 \times 4} + \cdots$$

If n is an integer, this expansion stops at b^n; if not, it forms an infinite series.

Since the expansion of any binomial can be obtained by substituting values for a, b, and n in the binomial formula, it is worth while, for some purposes, to commit the formula to memory. Generally, however, it is preferable to remember the procedure of expansion used in the preceding examples, rather than to memorize the formula.

Note that any desired term can be obtained from the binomial formula without writing the complete expansion. All that is

necessary to obtain any desired term, say, the sixth term, is to write the coefficient as $\frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \cdot \frac{(n-3)}{4} \cdot \frac{(n-4)}{5} \cdot \cdot \cdot$, continuing until the last number in the *denominator* is 5, or $6 - 1$. The exponent of a will be $n - 5$, and the exponent of b will be 5. Thus the sixth term is

$$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot a^{n-5}b^5$$

(Remember that this is the *sixth* term, *not* the fifth.)

In general, the rth term is

$$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \cdot \cdot \frac{n-r+2}{r-1} a^{n-r-1}b^{r-}$$

Example 1. Find the tenth term of $(x - y)^{12}$.

Solution: First write the denominator of the coefficient: $\frac{(\qquad\qquad)}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}$; then write n, $n - 1$, etc., in the numerator:

$$\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9} x^3(-y)^9$$

Observe that the exponent of x is $12 - 9 = 3$. Note also that most of the factors in the coefficient cancel out. Thus, the desired term is $4 \times 11 \times 5x^3(-y^9)$, or $-220x^3y^9$.

EXERCISES (142)

Determine the terms indicated.

1. 5th term of $(x + y)^9$
2. 4th term of $(c + d)^8$
3. 8th term of $(x + y)^{12}$
4. 3d term of $(a + b)^{20}$
5. 7th term of $(2 - x)^{11}$
6. 3d term of $(x + 1)^{40}$
7. 11th term of $(x + 1)^{15}$
8. 5th term of $(a - 2b)^7$
9. 5th term of $(1.02)^9$
10. 4th term of $(.995)^8$

143. Computation by the Binomial Theorem. In many cases expansion by means of the binomial theorem greatly reduces the labor involved in certain computations. Though the practical situations that necessitate such computations will not be presented here, the methods of computation themselves will be illustrated.

In order to facilitate application of the methods that follow, it is advisable to memorize the expansion

$$(1 + x)^n = 1 + nx + \frac{n}{1} \cdot \frac{n-1}{2} x^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + \cdots$$

This is the special case of the binomial formula in which $a = 1$ and $b = x$.

Example 1. Evaluate $\dfrac{1}{1.00016}$ to six figures.

Solution:
$$\frac{1}{1.00016} = (1 + .00016)^{-1}$$

We may now expand this by the methods already used in this chapter, or simply substitute in the general expansion of $(1 + x)^n$ above, with $x = .00016$ and $n = 1$.

Thus
$$\begin{aligned}(1 + .00016)^{-1} &= 1 + (-1)(.00016) + \frac{-1}{1} \cdot \frac{-2}{2} (.00016)^2 \\ &\quad + \frac{-1}{1} \cdot \frac{-2}{2} \cdot \frac{-3}{3} (.00016)^3 + \cdots \\ &= 1 - .00016 + 1.6^2 \times 10^{-8} + \cdots\end{aligned}$$

and there is no point in including more terms. Even the term $1.6^2 \times 10^{-8}$ need not be evaluated; for it will not influence the sixth figure in the result, which is $1 - .000160 = .999840$.

It is only when x is small in comparison with 1 that the expansion of $(1 + x)^n$ converges rapidly, *i.e.*, resolves into terms of negligible size after the first few.

Example 2. Evaluate $\dfrac{1}{\sqrt[3]{1.0024^2}}$.

Solution:
$$\begin{aligned}\frac{1}{\sqrt[3]{1.0024^2}} &= (1.0024)^{-\frac{2}{3}} = (1 + .0024)^{-\frac{2}{3}} \\ &= 1 - \tfrac{2}{3}(.0024) + \frac{-\frac{2}{3}}{1} \cdot \frac{-\frac{5}{3}}{2} (.0024)^2 + \cdots \\ &= 1 - .0016 + \tfrac{5}{9}(.0024)^2 + \cdots \\ &\doteq 1 - .0016 = .9984\end{aligned}$$

The method that has been described applies to binomials of the form $(1 + x)^n$, where x is much smaller than 1. It can also be used with binomials of the form $(a + b)^n$, if one of the numbers

a and b is much larger than the other, by dividing the binomial expression by a^n or b^n.

Example 3. Evaluate $(2.012)^5$.

Solution:
$$(2.012)^5 = 2^5\left(\frac{2.012}{2}\right)^5 = 2^5(1.006)^5 = 2^5(1 + .006)^5$$
$$= 2^5\left[1 + 5(.006) + \frac{5}{1}\cdot\frac{4}{2}(.006)^2 + \cdots\right.$$
$$= 2^5(1 + .03 + .00036 + \cdots)$$
$$\doteq 2^5(1.03036) \doteq 32 \times 1.030 = 32.96$$

The result would be 32.97 if the last two digits in 1.03036 were retained.

Example 4. Evaluate $\dfrac{1}{\sqrt[3]{8.0752}}$.

Solution:

$$\frac{1}{\sqrt[3]{8.0752}} = (8.0752)^{-\frac{1}{3}} = 8^{-\frac{1}{3}}(1.0094)^{-\frac{1}{3}} = \tfrac{1}{2}(1.0094)^{-\frac{1}{3}}$$
$$= \frac{1}{2}\left[1 - \frac{1}{3}(.0094) + \frac{-\frac{1}{3}}{1}\cdot\frac{-\frac{4}{3}}{2}(.0094)^2 + \cdots\right.$$
$$= \tfrac{1}{2}[1 - .00313 + \tfrac{2}{9}(.0094)^2 + \cdots]$$
$$\doteq \tfrac{1}{2}[1 - .00313 + .0000196 + \cdots] \doteq \frac{.99689}{2} \doteq .49844$$

One of the most important applications of the method is its use in evaluating the difference of two quantities that are almost equal.

Example 5. Evaluate $\dfrac{1}{1.00014} - \dfrac{1}{1.00016}$ to two figures.

Solution: To evaluate this expression to only two figures requires values of the two terms themselves accurate to six figures.

$$\frac{1}{1.00014} = (1 + .00014)^{-1} = 1 - .00014 + \frac{-1}{1}\cdot\frac{-2}{2}(.00014)^2 + \cdots$$
$$= 1 - .00014 + (.00014)^2 + \cdots$$
$$\frac{1}{1.00016} = (1 + .00016)^{-1} = 1 - .00016 + (.00016)^2 + \cdots$$

Only two terms are required in each expansion; for in each case the third term will not influence the sixth decimal place. Then

$$\frac{1}{1.00014} \doteq 1 - .000140 = .999860$$

and
$$\frac{1}{1.00016} \doteq 1 - .000160 = .999840$$

The difference is .000020, so that

$$\frac{1}{1.00014} - \frac{1}{1.00016} \doteq .000020$$

Sometimes the factor to be removed is not an integral power.

Example 6. Evaluate $\dfrac{1}{\sqrt{3.0171}} - \dfrac{1}{\sqrt{3.0174}}$.

Solution:

$$\begin{aligned}
\frac{1}{\sqrt{3.0171}} - \frac{1}{\sqrt{3.0174}} &= (3.0171)^{-\frac{1}{2}} - (3.0174)^{-\frac{1}{2}} \\
&= 3^{-\frac{1}{2}}\left[(1.0057)^{-\frac{1}{2}} - (1.0058)^{-\frac{1}{2}}\right] \\
&= \frac{1}{\sqrt{3}}\left[(1 - .00285 + \cdots) - (1 - .00290 + \cdots)\right] \\
&\doteq \frac{.00005}{\sqrt{3}}
\end{aligned}$$

which can now be evaluated by logarithms. This result could not be obtained by using four-place logarithms to evaluate the original terms.

EXERCISES (143)

Evaluate to five significant figures:

1. $\sqrt{1.00072}$
2. $\sqrt{1.0032}$
3. $\sqrt{4.0032}$
4. $\sqrt{9.0153}$
5. $\dfrac{1}{\sqrt{9.027}}$
6. $\dfrac{1}{\sqrt[3]{8.024}}$
7. $\sqrt[4]{16.032}$
8. $\dfrac{1}{\sqrt[3]{27.162}}$
9. $\sqrt{3.9984}$
10. $\sqrt[3]{7.998}$
11. $\dfrac{1}{\sqrt{8.9928}}$
12. $\dfrac{1}{\sqrt[4]{15.984}}$
13. $\dfrac{1}{1.00032} - \dfrac{1}{1.00035}$
14. $\dfrac{1}{\sqrt{1.00016}} - \dfrac{1}{\sqrt{1.00018}}$
15. $\dfrac{1}{4.00128} - \dfrac{1}{4.00164}$
16. $\dfrac{1}{\sqrt[3]{8.0034}} - \dfrac{1}{\sqrt[3]{8.0040}}$.

Table 1. Square Roots of Numbers

N	$\sqrt{N}$	$\sqrt{10N}$	N	$\sqrt{N}$	$\sqrt{10N}$
1	**1.000**	**3.162**	**51**	7.141	22.58
2	**1.414**	**4.472**	**52**	7.211	22.80
3	**1.732**	**5.477**	**53**	7.280	23.02
4	**2.000**	**6.325**	**54**	7.348	23.24
5	**2.236**	**7.071**	**55**	7.416	23.45
6	**2.449**	**7.746**	**56**	7.483	23.66
7	**2.646**	**8.367**	**57**	7.550	23.87
8	**2.828**	**8.944**	**58**	7.616	24.08
9	**3.000**	**9.487**	**59**	7.681	24.29
10	3.162	10.00	**60**	7.746	24.49
11	3.317	10.49	**61**	7.810	24.70
12	3.464	10.95	**62**	7.874	24.90
13	3.606	11.40	**63**	7.937	25.10
14	3.742	11.83	**64**	8.000	25.30
15	3.873	12.25	**65**	8.062	25.50
16	4.000	12.65	**66**	8.124	25.69
17	4.123	13.04	**67**	8.185	25.88
18	4.243	13.42	**68**	8.246	26.08
19	4.359	13.78	**69**	8.307	26.27
20	4.472	14.14	**70**	8.367	26.46
21	4.583	14.49	**71**	8.426	26.65
22	4.690	14.83	**72**	8.485	26.83
23	4.796	15.17	**73**	8.544	27.02
24	4.899	15.49	**74**	8.602	27.20
25	5.000	15.81	**75**	8.660	27.39
26	5.099	16.12	**76**	8.718	27.57
27	5.196	16.43	**77**	8.775	27.75
28	5.292	16.73	**78**	8.832	27.93
29	5.385	17.03	**79**	8.888	28.11
30	5.477	17.32	**80**	8.944	28.28
31	5.568	17.61	**81**	9.000	28.46
32	5.657	17.89	**82**	9.055	28.64
33	5.745	18.17	**83**	9.110	28.81
34	5.831	18.44	**84**	9.165	28.98
35	5.916	18.71	**85**	9.220	29.15
36	6.000	18.97	**86**	9.274	29.33
37	6.083	19.24	**87**	9.327	29.50
38	6.164	19.49	**88**	9.381	29.66
39	6.245	19.75	**89**	9.434	29.83
40	6.325	20.00	**90**	9.487	30.00
41	6.403	20.25	**91**	9.539	30.17
42	6.481	20.49	**92**	9.592	30.33
43	6.557	20.74	**93**	9.644	30.50
44	6.633	20.98	**94**	9.695	30.66
45	6.708	21.21	**95**	9.747	30.82
46	6.782	21.45	**96**	9.798	30.98
47	6.856	21.68	**97**	9.849	31.14
48	6.928	21.91	**98**	9.899	31.30
49	7.000	22.14	**99**	9.950	31.46
50	7.071	22.36	**100**	10.00	31.62

Table 2. Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	7522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6876	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

Table 2. Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Table 3. Trigonometric Functions

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
0° 00′	.0000		1.0000	0.0000	.0000				90° 00′
10′	.0029	7.4637	1.0000	0000	.0029	7.4637	343.8	2.5363	50
20′	.0058	7648	1.0000	0000	.0058	7648	171.9	2352	40′
30′	.0087	9408	1.0000	0000	.0087	9409	114.6	0591	30′
40′	.0116	8.0658	.9999	0000	.0116	8.0658	85.94	1.9342	20′
50′	.0145	1627	.9999	0000	.0145	1627	68.75	8373	10′
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.29	1.7581	89° 00′
10′	.0204	3088	.9998	9999	.0204	3089	49.10	6911	50′
20′	.0233	3668	.9997	9999	.0233	3669	42.96	6331	40′
30′	.0262	4179	.9997	9999	.0262	4181	38.19	5819	30′
40′	.0291	4637	.9996	9998	.0291	4638	34.37	5362	20′
50′	.0320	5050	.9995	9998	.0320	5053	31.24	4947	10′
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.64	1.4569	88° 00′
10′	.0378	5776	.9993	9997	.0378	5779	26.43	4221	50′
20′	.0407	6097	.9992	9996	.0407	6101	24.54	3899	40′
30′	.0436	6397	.9990	9996	.0437	6401	22.90	3599	30′
40′	.0465	6677	.9989	9995	.0466	6682	21.47	3318	20′
50′	.0494	6940	.9988	9995	.0495	6945	20.21	3055	10′
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.08	1.2806	87° 00′
10′	.0552	7423	.9985	9993	.0553	7429	18.08	2571	50′
20′	.0581	7645	.9983	9993	.0582	7652	17.17	2348	40′
30′	.0610	7857	.9981	9992	.0612	7865	16.35	2135	30′
40′	.0640	8059	.9980	9991	.0641	8067	15.60	1933	20′
50′	.0669	8251	.9978	9990	.0670	8261	14.92	1739	10′
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.30	1.1554	86° 00′
10′	.0727	8613	.9974	9989	.0729	8624	13.73	1376	50′
20′	.0756	8783	.9971	9988	.0758	8795	13.20	1205	40′
30′	.0785	8946	.9969	9987	.0787	8960	12.71	1040	30′
40′	.0814	9104	.9967	9986	.0816	9118	12.25	0882	20′
50′	.0843	9256	.9964	9985	.0846	9272	11.83	0728	10′
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.43	1.0580	85° 00′
10′	.0901	9545	.9959	9982	.0904	9563	11.06	0437	50′
20′	.0929	9682	.9957	9981	.0934	9701	10.71	0299	40′
30′	.0958	9816	.9954	9980	.0963	9836	10.38	0164	30′
40′	.0987	9945	.9951	9979	.0992	9966	10.08	0034	20′
50′	.1016	9.0070	.9948	9977	.1022	9.0093	9.788	0.9907	10′
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.514	0.9784	84° 00′
10′	.1074	0311	.9942	9975	.1080	0336	9.255	9664	50′
20′	.1103	0426	.9939	9973	.1110	0453	9.010	9547	40′
30′	.1132	0539	.9936	9972	.1139	0567	8.777	9433	30′
40′	.1161	0648	.9932	9971	.1169	0678	8.556	9322	20′
50′	.1190	0755	.9929	9969	.1198	0786	8.345	9214	10′
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.144	0.9109	83° 00′
10′	.1248	0961	.9922	9966	.1257	0995	7.953	9005	50′
20′	.1276	1060	.9918	9964	.1287	1096	7.770	8904	40′
30′	.1305	1157	.9914	9963	.1317	1194	7.596	8806	30′
40′	.1334	1252	.9911	9961	.1346	1291	7.429	8709	20′
50′	.1363	1345	.9907	9959	.1376	1385	7.269	8615	10′
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.115	0.8522	82° 00′
10′	.1421	1525	.9899	9956	.1435	1569	6.968	8431	50′
20′	.1449	1612	.9894	9954	.1465	1658	6.827	8342	40′
30′	.1478	1697	.9890	9952	.1495	1745	6.691	8255	30′
40′	.1507	1781	.9886	9950	.1524	1831	6.561	8169	20′
50′	.1536	1863	.9881	9948	.1554	1915	6.435	8085	10′
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.314	0.8003	81° 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

Table 3. Trigonometric Functions

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.314	0.8003	81° 00′
10′	.1593	2022	.9872	9944	.1614	2078	6.197	7922	50′
20′	.1622	2100	.9868	9942	.1644	2158	6.084	7842	40′
30′	.1650	2176	.9863	9940	.1673	2236	5.976	7764	30′
40′	.1679	2251	.9858	9938	.1703	2313	5.871	7687	20′
50′	.1708	2324	.9853	9936	.1733	2389	5.769	7611	10′
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.671	0.7537	80° 00′
10′	.1765	2468	.9843	9931	.1793	2536	5.576	7464	50′
20′	.1794	2538	.9838	9929	.1823	2609	5.484	7391	40′
30′	.1822	2606	.9833	9927	.1853	2680	5.396	7320	30′
40′	.1851	2674	.9827	9924	.1883	2750	5.309	7250	20′
50′	.1880	2740	.9822	9922	.1914	2819	5.226	7181	10′
11° 00′	.1908	9.2806	.9816	9.9919	.1974	9.2887	5.145	0.7113	79° 00′
10′	.1937	2870	.9811	9917	.1974	2953	5.066	7047	50′
20′	.1965	2934	.9805	9914	.2004	3020	4.989	6980	40′
30′	.1994	2997	.9799	9912	.2035	3085	4.915	6915	30′
40′	.2022	3058	.9793	9909	.2065	3149	4.843	6851	20′
50′	.2051	3119	.9787	9907	.2095	3212	4.773	6788	10′
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.705	0.6725	78° 00′
10′	.2108	3238	.9775	9901	.2156	3336	4.638	6664	50′
20′	.2136	3296	.9769	9899	.2186	3397	4.574	6603	40′
30′	.2164	3353	.9763	9896	.2217	3458	4.511	6542	30′
40′	.2193	3410	.9757	9893	.2247	3517	4.449	6483	20′
50′	.2221	3466	.9750	9890	.2278	3576	4.390	6424	10′
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.332	0.6366	77° 00′
10′	.2278	3575	.9737	9884	.2339	3691	4.275	6309	50′
20′	.2306	3629	.9730	9881	.2370	3748	4.219	6252	40′
30′	.2334	3682	.9724	9878	.2401	3804	4.165	6196	30′
40′	.2363	3734	.9717	9875	.2432	3859	4.113	6141	20′
50′	.2391	3786	.9710	9872	.2462	3914	4.061	6086	10′
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.011	0.6032	76° 00′
10′	.2447	3887	.9696	9866	.2524	4021	3.962	5979	50′
20′	.2476	3937	.9689	9863	.2555	4074	3.914	5926	40′
30′	.2504	3986	.9681	9859	.2586	4127	3.867	5873	30′
40′	.2532	4035	.9674	9856	.2617	4178	3.821	5822	20′
50′	.2560	4083	.9667	9853	.2648	4230	3.776	5770	10′
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.732	0.5719	75° 00′
10′	.2616	4177	.9652	9846	.2711	4331	3.689	5669	50′
20′	.2644	4223	.9644	9843	.2742	4381	3.647	5619	40′
30′	.2672	4269	.9636	9839	.2773	4430	3.606	5570	30′
40′	.2700	4314	.9628	9836	.2805	4479	3.566	5521	20′
50′	.2728	4359	.9621	9832	.2836	4527	3.526	5473	10′
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.487	0.5425	74° 00′
10′	.2784	4447	.9605	9825	.2899	4622	3.450	5378	50′
20′	.2812	4491	.9596	9821	.2931	4669	3.412	5331	40′
30′	.2840	4533	.9588	9817	.2962	4716	3.376	5284	30′
40′	.2868	4576	.9580	9814	.2994	4762	3.340	5238	20′
50′	.2896	4618	.9572	9810	.3026	4808	3.305	5192	10′
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.271	0.5147	73° 00′
10′	.2952	4700	.9555	9802	.3089	4898	3.237	5102	50′
20′	.2979	4741	.9546	9798	.3121	4943	3.204	5057	40′
30′	.3007	4781	.9537	9794	.3153	4987	3.172	5013	30′
40′	.3035	4821	.9528	9790	.3185	5031	3.140	4969	20′
50′	.3062	4861	.9520	9786	.3217	5075	3.108	4925	10′
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.078	0.4882	72° 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

Table 3. Trigonometric Functions

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.078	0.4882	**72°** 00′
10′	.3118	4939	.9502	9778	.3281	5161	3.048	4839	50′
20′	.3145	4977	.9492	9774	.3314	5203	3.018	4797	40′
30′	.3173	5015	.9483	9770	.3346	5245	2.989	4755	30′
40′	.3201	5052	.9474	9765	.3378	5287	2.960	4713	20′
50′	.3228	5090	.9465	9761	.3411	5329	2.932	4671	10′
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.904	0.4630	**71°** 00′
10′	.3283	5163	.9446	9752	.3476	5411	2.877	4589	50′
20′	.3311	5199	.9436	9748	.3508	5451	2.850	4549	40′
30′	.3338	5235	.9426	9743	.3541	5491	2.824	4509	30′
40′	.3365	5270	.9417	9739	.3574	5531	2.798	4469	20′
50′	.3393	5306	.9407	9734	.3607	5571	2.772	4429	10′
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.748	0.4389	**70°** 00′
10′	.3448	5375	.9387	9725	.3673	5650	2.723	4350	50′
20′	.3475	5409	.9377	9721	.3706	5689	2.698	4311	40′
30′	.3502	5443	.9367	9716	.3739	5727	2 675	4273	30′
40′	.3529	5477	.9356	9711	.3772	5766	2.651	4234	20′
50′	.3557	5510	.9346	9706	.3805	5804	2.628	4196	10′
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.605	0.4158	**69°** 00′
10	.3611	5576	.9325	9697	.3872	5879	2.583	4121	50′
20′	.3638	5609	.9315	9692	.3906	5917	2.560	4083	40′
30′	.3665	5641	.9304	9687	.3939	5954	2.539	4046	30′
40′	.3692	5673	.9293	9682	.3973	5991	2.517	4009	20′
50′	.3719	5704	.9283	9677	.4006	6028	2.496	3972	10′
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.475	0.3936	**68°** 00′
10′	.3773	5767	.9261	9667	.4074	6100	2.454	3900	50′
20′	.3800	5798	.9250	9661	.4108	6136	2.434	3864	40′
30′	.3827	5828	.9239	9656	.4142	6172	2.414	3828	30′
40′	.3854	5859	.9228	9651	.4176	6208	2.394	3792	20′
50′	.3881	5889	.9216	9646	.4210	6243	2.375	3757	10′
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.356	0.3721	**67°** 00′
10′	.3934	5948	.9194	9635	.4279	6314	2.337	3686	50′
20′	.3961	5978	.9182	9629	.4314	6348	2.318	3652	40′
30′	.3987	6007	.9171	9624	.4348	6383	2.300	3617	30′
40′	.4014	6036	.9159	9618	.4383	6417	2.282	3583	20′
50′	.4041	6065	.9147	9613	.4417	6452	2.264	3548	10′
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.246	0.3514	**66°** 00′
10′	.4094	6121	.9124	9602	.4487	6520	2.229	3480	50′
20′	.4120	6149	.9112	9596	.4522	6553	2.211	3447	40′
30′	.4147	6177	.9100	9590	.4557	6587	2.194	3413	30′
40′	.4173	6205	.9088	9584	.4592	6620	2.178	3380	20′
50′	.4200	6232	.9075	9579	.4628	6654	2.161	3346	10′
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.144	0.3313	**65°** 00′
10′	.4253	6286	.9051	9567	.4699	6720	2.128	3280	50′
20′	.4279	6313	.9038	9561	.4734	6752	2.112	3248	40′
30′	.4305	6340	.9026	9555	.4770	6785	2.096	3215	30′
40′	.4331	6366	.9013	9549	.4806	6817	2.081	3183	20′
50′	.4358	6392	.9001	9543	.4841	6850	2.066	3150	10′
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.050	0.3118	**64°** 00′
10′	.4410	6444	.8975	9530	.4913	6914	2.035	3086	50′
20′	.4436	6470	.8962	9524	.4950	6946	2.020	3054	40′
30′	.4462	6495	.8949	9518	.4986	6977	2.006	3023	30′
40′	.4488	6521	.8936	9512	.5022	7009	1.991	2991	20′
50′	.4514	6546	.8923	9505	.5059	7040	1.977	2960	10′
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.963	0.2928	**63°** 00′
Angle	**cos**	**log cos**	**sine**	**log sin**			**tan**	**log tan**	**Angle**

Table 3. Trigonometric Functions

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.963	0.2928	63° 00′
10′	.4566	6595	.8897	9492	.5132	7103	1.949	2897	50′
20′	.4592	6620	.8884	9486	.5169	7134	1.935	2866	40′
30′	.4617	6644	.8870	9479	.5206	7165	1.921	2835	30′
40′	.4643	6668	.8857	9473	.5243	7196	1.907	2804	20′
50′	.4669	6692	.8843	9466	.5280	7226	1.894	2774	10′
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.881	0.2743	62° 00′
10′	.4720	6740	.8816	9453	.5354	7287	1.868	2713	50′
20′	.4746	6763	.8802	9446	.5392	7317	1.855	2683	40′
30′	.4772	6787	.8788	9439	.5430	7348	1.842	2652	30′
40′	.4797	6810	.8774	9432	.5467	7378	1.829	2622	20′
50′	.4823	6833	.8760	9425	.5505	7408	1.816	2592	10′
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.804	0.2562	61° 00′
10′	.4874	6878	.8732	9411	.5581	7467	1.792	2533	50′
20′	.4899	6901	.8718	9404	.5619	7497	1.780	2503	40′
30′	.4924	6923	.8704	9397	.5658	7526	1.768	2474	30′
40′	.4950	6946	.8689	9390	.5696	7556	1.756	2444	20′
50′	.4975	6968	.8675	9383	.5735	7585	1.744	2415	10′
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.732	0.2386	60° 00′
10′	.5025	7012	.8646	9368	.5812	7644	1.720	2356	50′
20′	.5050	7033	.8631	9361	.5851	7673	1.709	2327	40′
30′	.5075	7055	.8616	9353	5890	7701	1.698	2299	30′
40′	.5100	7076	.8601	9346	.5930	7730	1.686	2270	20′
50′	.5125	7097	.8587	9338	.5969	7759	1 675	2241	10′
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.664	0.2212	59° 00′
10′	.5175	7139	.8557	9323	.6048	7816	1.653	2184	50′
20′	.5200	7160	.8542	9315	.6088	7845	1.643	2155	40′
30′	.5225	7181	.8526	9308	.6128	7873	1.632	2127	30′
40′	.5250	7201	.8511	9300	.6168	7902	1.621	2098	20′
50′	.5275	7222	.8496	9292	.6208	7930	1.611	2070	10′
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1 600	0.2042	58° 00′
10′	.5324	7262	.8465	9276	.6289	7986	1.590	2014	50′
20′	.5348	7282	.8450	9268	.6330	8014	1.580	1986	40′
30′	.5373	7302	.8434	9260	.6371	8042	1.570	1958	30′
40′	.5398	7322	.8418	9252	.6412	8070	1.560	1930	20′
50′	.5422	7342	.8403	9244	.6453	8097	1.550	1903	10′
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.540	0.1875	57° 00′
10′	.5471	7380	.8371	9228	.6536	8153	1.530	1847	50′
20′	.5495	7400	.8355	9219	.6577	8180	1.520	1820	40′
30′	.5519	7419	.8339	9211	.6619	8208	1.511	1792	30′
40′	.5544	7438	.8323	9203	.6661	8235	1.501	1765	20′
50′	.5568	7457	.8307	9194	.6703	8263	1.492	1737	10′
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.483	0.1710	56° 00′
10′	.5616	7494	.8274	9177	.6787	8317	1.473	1683	50′
20′	.5640	7513	.8258	9169	.6830	8344	1.464	1656	40′
30′	.5664	7531	.8241	9160	.6873	8371	1.455	1629	30′
40′	.5688	7550	.8225	9151	.6916	8398	1.446	1602	20′
50′	.5712	7568	.8208	9142	.6959	8425	1.437	1575	10′
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.428	0.1548	55° 00′
10′	.5760	7604	.8175	9125	.7046	8479	1.419	1521	50′
20′	.5783	7622	.8158	9116	.7089	8506	1.411	1494	40′
30′	.5807	7640	.8141	9107	.7133	8533	1.402	1467	30′
40′	.5831	7657	.8124	9098	.7177	8559	1.393	1441	20′
50′	.5854	7675	.8107	9089	.7221	8586	1.385	1414	10′
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.376	0.1387	54° 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

Table 3. Trigonometric Functions

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.376	0.1387	**54°** 00′
10′	.5901	7710	.8073	9070	.7310	8639	1.368	1361	50′
20′	.5925	7727	.8056	9061	.7355	8666	1.360	1334	40′
30′	.5948	7744	.8039	9052	.7400	8692	1.351	1308	30′
40′	.5972	7761	.8021	9042	.7445	8718	1.343	1282	20′
50′	.5995	7778	.8004	9033	.7490	8745	1.335	1255	10′
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.327	0.1229	**53°** 00′
10′	.6041	7811	.7969	9014	.7581	8797	1.319	1203	50′
20′	.6065	7828	.7951	9004	.7627	8824	1.311	1176	40′
30′	.6088	7844	.7934	8995	.7673	8850	1.303	1150	30′
40′	.6111	7861	.7916	8985	.7720	8876	1.295	1124	20′
50′	.6134	7877	.7898	8975	.7766	8902	1.288	1098	10′
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.280	0.1072	**52°** 00′
10′	.6180	7910	.7862	8955	.7860	8954	1.272	1046	50′
20′	.6202	7926	.7844	8945	.7907	8980	1.265	1020	40′
30′	.6225	7941	.7826	8935	.7954	9006	1.257	0994	30′
40′	.6248	7957	.7808	8925	.8002	9032	1.250	0968	20′
50′	.6271	7973	.7790	8915	.8050	9058	1.242	0942	10′
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.235	0.0916	**51°** 00′
10′	.6316	8004	.7753	8895	.8146	9110	1.228	0890	50′
20′	.6338	8020	.7735	8884	.8195	9135	1.220	0865	40′
30′	.6361	8035	.7716	8874	.8243	9161	1.213	0839	30′
40′	.6383	8050	.7698	8864	.8292	9187	1.206	0813	20′
50′	.6406	8066	.7679	8853	.8342	9212	1.199	0788	10′
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.192	0.0762	**50°** 00′
10′	.6450	8096	.7642	8832	.8441	9264	1.185	0736	50′
20′	.6472	8111	.7623	8821	.8491	9289	1.178	0711	40′
30′	.6494	8125	.7604	8810	.8541	9315	1.171	0685	30′
40′	.6517	8140	.7585	8800	.8591	9341	1.164	0659	20′
50′	.6539	8155	.7566	8789	.8642	9366	1.157	0634	10′
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.150	0.0608	**49°** 00′
10′	.6583	8184	.7528	8767	.8744	9417	1.144	0583	50′
20′	.6604	8198	.7509	8756	.8796	9443	1.137	0557	40′
30′	.6626	8213	.7490	8745	.8847	9468	1.130	0532	30′
40′	.6648	8227	.7470	8733	.8899	9494	1.124	0506	20′
50′	.6670	8241	.7451	8722	.8952	9519	1.117	0481	10′
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.111	0.0456	**48°** 00′
10′	.6713	8269	.7412	8699	.9057	9570	1.104	0430	50′
20′	.6734	8283	.7392	8688	.9110	9595	1.098	0405	40′
30′	.6756	8297	.7373	8676	.9163	9621	1.091	0379	30′
40′	.6777	8311	.7353	8665	.9217	9646	1.085	0354	20′
50′	.6799	8324	.7333	8653	.9271	9671	1.079	0329	10′
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.072	0.0303	**47°** 00′
10′	.6841	8351	.7294	8629	.9380	9722	1.066	0278	50′
20′	.6862	8365	.7274	8618	.9435	9747	1.060	0253	40′
30′	.6884	8378	.7254	8606	.9490	9772	1.054	0228	30′
40′	.6905	8391	.7234	8594	.9545	9798	1.048	0202	20′
50′	.6926	8405	.7214	8582	.9601	9823	1.042	0177	10′
44° 00′	.6947	9.8418	.7319	9.8569	.9557	9.9848	1.036	0.0152	**46°** 00′
10′	.6967	8431	.7173	8557	.9713	9874	1.030	0126	50′
20′	.6988	8444	.7153	8545	.9770	9899	1.024	0101	40′
30′	.7009	8457	.7133	8532	.9827	9924	1.018	0076	30′
40′	.7030	8469	.7112	8520	.9884	9949	1.012	0051	20′
50′	.7050	8482	.7092	8507	.9942	9975	1.006	0025	10′
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.000	0.0000	**45°** 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

ANSWERS

Section 7, Page 20

1. $N_1 + N_2 = 60$
   $N_1 - N_2 = 14$
3. $5N + 10D = 395$
   $N + D = 55$
5. $1{,}000 = 5(225 - S_w)$
7. $x + y = 1{,}500$
   $y = x + 135$
9. $.02x + .03y = 275$
   $x + y = 10{,}000$
11. $.04x + .03y = 351$
    $x = y + 2000$
13. $8S_1 = 6S_2$
    $S_2 = S_1 + 45$
15. $50x = 20 + 40y$
    $y = x + 3$

Section 9, Page 24

1. $28\frac{2}{7}$ sq. ft.
3. 144 ft.
5. 24 sq. ft.
7. \$459
9. $N_1 + N_2 = S$
   $N_1 = 2N_2$
11. $A_1P_1 + A_2P_2 = MP_3$
    $A_1 + A_2 = M$
13. $(S_1 + S_2)T = D$
    $S_1 = 2S_2$
15. $M_1 - M_2 = D$
    $M_1R_1 + M_2R_2 = I$

Section 17, Page 33

1. -96 3. 53 5. -27 7. -21 9. -38 11. -166
13. -40 15. 16 17. 379 19. 5 21. -15 23. 14
25. -16 27. -49 29. 16 31. 28 33. 20 35. 85
37. -33 39. -131

Section 18, Page 34

1. 40 3. -72 5. 12 7. 105 9. -72 11. -90 13. -3
15. -2 17. 29 19. -5 21. 6 23. 2 25. 2 27. -1

Section 19, Page 35

1. -29 3. 0 5. 1 7. 58 9. -2 11. -2 13. -1
15. 3

Section 20, Page 37

1. $14a$
3. $5ax$
5. $9a + 9b + 14c$
7. $15x^2 + 11xy$
9. $a^2 + 2a + 7$
11. $x^2 + y^2$
13. $3x^2 + 6x - 3 + y^2$
15. $4x + 8y$
17. $6a^2 - 3b^2$
19. $2a - b + 2c$
21. $2m^2 - 2mn - 6$
23. $7a^3 + 2a^2 - 7a - 2$
25. $\frac{1}{6}x + \frac{5}{6}y$
27. $-3.16a^2 + 3.27ab + .08b^2$

Section 21, Page 39

1. $5x - 6y$
3. $-4a + b$
5. $2a$
7. $2x^2 - 4x + 3$
9. $-5a$
11. $-9x^2 - 5x - 13$

Section 22, Page 40

1. $12x$
3. $6x^2$
5. $-36ab^2$
7. $100a^3b^5c^5$
9. $-6x^5y^9z^7$
11. $24a^3b^6c^4$
13. $24x^{a+5}y^{b+4}$

Section 23, Page 40

1. $20x - 15y$ **3.** $-21a - 18$ **5.** $6a^3b^2 + 8a^2b^3$
7. $-3x^2y^2 + 6y^3$ **9.** $-15a^3b^2 + 30ab^3 - 10a^2b^3 + 35ab^2$
11. $2x^{n+2} + 3x^ny^2$

Section 24, Page 41

1. $12a^2 + 11a - 15$ **3.** $4m^2 - 14mn + 6n^2$
5. $12x^4 - 10x^2y^2 + 2y^4$ **7.** $-m^3n + 7m^2n + 5mn - 35n$
9. $28m^3 - 8m^2n - 21mn + 6n^2$ **11.** $12a^2 + 14a - 16ab - 7b + 5b^2$
13. $6x^2 + 2x - 38xy - 8y + 56y^2$ **15.** $15x^5 - 8x^4 + 20x^3 - 25x^2 + 10x - 12$

Section 25, Page 43

1. $-8x^5$ **3.** $7a^2b^2$ **5.** $-29a^2b^2c^2$ **7.** $2x^{m-3}y^{n-2}$
9. $2x^{m+2}y^n$ **11.** $2x^{m-n+1}y^{n-m+1}$

Section 26, Page 43

1. $2a^2 - 4a$ **3.** $-4b + 7a$ **5.** $a^2bc^2 - 2ab^2c - c^2$
7. $-3abc + 4a^2b + 9bc^2$ **9.** $-22b + 5a - 9ab^2$ **11.** $8x^3 - 17x^2 - 27x$

Section 27, Page 46

1. $x - 2$ **3.** $3a - 2b$ **5.** $y + 2$ **7.** $x^2 + 11xy + 24y^2$
9. $2y^2 + y - 1$ **11.** $x^3 - 8y^3$ **13.** $m - 2 + \dfrac{1}{m - 5}$
15. $3x - 1 + \dfrac{12}{x - 2}$ **17.** $3x^2 - 8x + 8 - \dfrac{6}{x + 1}$ **19.** $x + 4y + \dfrac{35y^2}{x - 7y}$

Section 30, Page 49

1. $x = 6$ **3.** $y = -5$ **5.** $x = 3$ **7.** $a = -3$
9. $x = \frac{1}{4}$ **11.** $x = 8$ **13.** $y = 3\frac{4}{7}$ **15.** $x = 3$
17. $2\frac{34}{79}$ **19.** $x = 3a$ **21.** $x = 11a - 12$ **23.** $x = -3a^2 + 6a - 5$

Section 33, Page 53

1. $x = 2$, $y = -2$ **3.** $x = 4$, $y = 5$ **5.** $x = 3\frac{2}{5}$, $y = -3\frac{1}{5}$
7. $m = 17\frac{1}{5}$, $n = 43\frac{1}{5}$ **9.** $x = 11\frac{7}{11}$, $y = 4\frac{8}{11}$ **11.** $x = 0$, $y = -5$

Section 34, Page 55

1. $m = 3$, $n = \frac{2}{5}$ **3.** $x = -4$, $y = 5$ **5.** $x = 3\frac{2}{5}$, $y = -3\frac{1}{5}$ **7.** $x = 17\frac{1}{5}$, $y = 43\frac{1}{5}$
9. $u = 11\frac{7}{11}$, $v = 4\frac{8}{11}$ **11.** $x = -14\frac{5}{9}$, $y = -11\frac{1}{3}$ **13.** $x = 1$, $y = 2$ **15.** $x = 2$, $y = 1\frac{4}{5}$
17. $x = 3$, $y = 1$ **19.** $x = 5\frac{7}{13}$, $y = 5\frac{1}{13}$ **21.** $x = 20$, $y = 32$ **23.** $x = 4\frac{4}{5}a$, $y = 1\frac{4}{5}a$
25. $x = \dfrac{2a + 3b}{13}$, $y = \dfrac{3a - 2b}{13}$ **27.** $x = \dfrac{a + b}{2m}$, $y = \dfrac{a - b}{2n}$

Section 35, Page 59

1. $x = 1$, $y = 1$, $z = 1$
3. $x = -10$, $y = 20$, $z = -8$
5. $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$
7. $x = 1$, $y = 2$, $z = 3$
9. $x = \dfrac{1}{2a}$, $y = \dfrac{1}{2b}$, $z = \dfrac{1}{2c}$

Section 36, Page 60

1. 37, 23
3. 31 nickels, 24 dimes
5. 25 m.p.h.
7. 682.5, 817.5 miles
9. $2,500 at 2%, $7,500 at 3%
11. $3,900 at 3%, $5,900 at 4%
13. 135 m.p.h., 180 m.p.h.
15. 14 hr., 17 hr.
17. $\dfrac{S + D}{2}, \dfrac{S - D}{2}$
19. $\dfrac{D}{2T} - \dfrac{M}{2}, \dfrac{D}{2T} + \dfrac{M}{2}$
21. 79°, 64°, 37°

Section 37, Page 63

1. 24.5, 13.5
3. 17 quarters, 23 dimes
5. $378.50, $191.50
7. 10 lb. at 50 cents, 15 lb. at 40 cents
9. $6,000 at 10%, $10,000 at 5%
11. 48 m.p.h., 52 m.p.h.
13. Walked $10\frac{1}{2}$ miles, rode 45 miles
15. 48, 50, 52

Section 40, Page 67

A

1. Horses, $212; mules, $180
3. 1,200 reserved, 3,600 general admission
5. 3 doz. spoiled
7. $.50, $.75, $6.50
9. 45 three-cent, 15 five-cent, 30 eight-cent

B

11. 43 years, 19 years
13. 32 years, 16 years
15. 24 years, 8 years

C

17. 140 m.p.h., 220 m.p.h.
19. $233\frac{1}{3}$ miles at 35 m.p.h., $166\frac{2}{3}$ at 50
21. 30 m.p.h.
23. 30 minutes, 120 miles
25. 280 m.p.h., 40 m.p.h.
27. 225 miles
29. 1500 miles

D

31. 75 gr. 50%, 125 gr. 90%
33. $11\frac{7}{13}$ oz.
35. 15 lb. for 65-cent mixture, 75-cent mixture not possible
37. 80% and 30%

E

39. $5\frac{1}{7}$ ft.
41. $1\frac{3}{7}$ ft.

F

43. $3,600 at 5%, $6,000 at 3%
45. 2% and 3%

G

47. 52 **49.** 4 and 7 **51.** 376

Section 45, Page 83

1. (1, 4) **3.** (5, 8) **5.** (2, 3) **7.** (5, 3) **9.** (0, −5)
11. Inconsistent **13.** $(3, \frac{1}{3})$ **15.** (3.6, −1.4) **17.** (−1.3, 2.7)

Section 48, Page 91

11. $z = 1, z = 4$ **13.** $x = 0, x = 3$ **15.** $w = 0, w = 1, w = -1$

Section 50, Page 97

1. Approximately 1 hr. 20 min. (accurately, $78\frac{3}{4}$ min.)
3. 16.8 hr. **5.** 4 hr. **7.** 21 hr. (approx.)

Section 51, Page 99

1. Minimum value, 1, for $x = 2$ **3.** $\frac{1}{8}$ sq. mile **5.** $\frac{1}{2}$ **7.** $3\frac{1}{8}$ sec.

Section 52, Page 101

1. $V(l) = \frac{1}{4}\pi d^2 l$ **3.** $V(d, l) = \frac{1}{4}\pi d^2 l$ **5.** $z(1, 3) = 10$, $z(4, 0) = 19$, $z(3, -5) = -18$

7. $z(0) = 7$, $z(2) = 9$, $z(a) = 2a^2 - 3a + 7$ **9.** $b(a) = 3a - 7$ **11.** $y(x) = \dfrac{25}{x}$

Section 55, Page 105

1. $3(x + 5)$ **3.** $2x(2x + y)$ **5.** $a(x^2 - y^2 + axy)$
7. $3(x^2 - 2xy + 3y^2)$ **9.** $x^2(3x^2 + 7x - 12)$

Section 56, Page 105

1. $(x + 3)(x + y)$ **3.** $(a - b)(4 - m)$ **5.** $(a^2 + b^2)(a - 1)$
7. $(5a^2 - 1)(a + 1)$ **9.** $(x^2 + 4)(3 - x)$ **11.** $(4x^2 + 3)(3 - 2x)$

Section 57, Page 108

1. $x^2 + 5x - 6$ **3.** $a^2 - a - 20$ **5.** $2x^2 + 9x + 4$
7. $x^2 + 7x + 12$ **9.** $x^2 - 6x - 7$ **11.** $6x^2 + 19x + 10$
13. $(x + 2)(x + 3)$ **15.** $(a + 8)(a + 1)$ **17.** $(x + 7)(x - 4)$
19. $(x + 1)(x - 21)$ **21.** $(x - 26)(x + 10)$ **23.** $(x - 6y)(x - y)$
25. $(2x + 1)(x + 1)$ **27.** $5(x + 3)(x + 2)$ **29.** $(6x - 1)(x - 1)$
31. $(3x + 1)(x + 3)$ **33.** $(3a - 1)(a + 3)$ **35.** $(2x + 7)(x - 4)$
37. $(6x - 7)(x + 1)$ **39.** $(3 - 4a)(2 - a)$ **41.** $(7a - 12b)(3a + 2b)$

Section 58, Page 109

1. $9x^2 - 25$ **3.** $36 - 25a^2$ **5.** $x^6 - 1$
7. $4x^6 - 9y^6$ **9.** $(a - 5)(a + 5)$ **11.** $(3x - 7y)(3x + 7y)$
13. $(7x - 5y)(7x + 5y)$ **15.** $(1 - 10xy)(1 + 10xy)$
17. $(.1x - .1y)(.1x + .1y)$ **19.** $(1 - 8m^6n^8)(1 + 8m^6n^8)$

Section 59, Page 110

1. $(x + y - z)(x + y + z)$
3. $(x - y + z)(x + y - z)$
5. $(a + b - c + d)(a + b + c - d)$
7. $(x + y - a - b)(x + y + a + b)$

Section 60, Page 111

1. $x = 2, x = 3$
3. $x = -4, x = -3$
5. $x = 0, x = a$
7. $x = 0, x = 2, x = 3$
9. $x = \frac{1}{2}, x = -\frac{1}{3}$
11. $x = \pm 6$
13. $x = 2a$
15. $x = \pm \frac{5}{4}a$

Section 61, Page 113

7. $(x + 2)(x^2 - 2x + 5)$
9. $(z - 3)(z^2 + z + 4)$
11. $(x + 1)(x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$
13. $(x^4 + y^4)(x^8 - x^4y^4 + y^8)$
15. $(x + 3)(x - 5)(x + 1)$
17. $(x - 3)(x + 2)(x^2 + 2x + 7)$
19. $(r^3 - s^3)(r^3 + s^3)(r^6 + s^6)$
21. $(b - 2)(b + 1)^2$
23. $(x + 2)(x^3 - 9x^2 + 19x - 35)$
25. $(a + b)^3$
27. $(x^2 + y^2)(x - y)(x + y)$
29. $(x - 1)(x^2 + x + 1)(x^{12} + x^9 + x^6 + x^3 + 1)$

or

$(x - 1)(x^4 + x^3 + x^2 + x + 1)(x^{10} + x^5 + 1)$

Section 62, Page 114

1. $3x(x + 1)^2$
3. $x^3y^4(6xy + 2 + 3x^4y^2)$
5. $(m + y)(m - x)$
7. $(2a - 3b)(2a + 3b)(a + 2b)$
9. $(a + 11)(a - 7)$
11. $b(2a + b)$
13. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$
15. $(x - 3)(x - y)$
17. $(x + 1)(x - 1)(y + 1)(y - 1)$
19. $(m - 1)(m + 1)(m + 2)(m + 3)$
21. $(n - 2)(n - 1)(n + 3)$
23. $(a - 2)(a + 1)(a^2 - a + 2)$
25. $2x(x - 2)(x + 2)(x + 3)$
27. $x(2xy + 1)(2xy - 1)(4x^2y^2 + 1)$
29. $(x + y - 2)(x - y)$

Section 63, Page 117

1. $\frac{1}{3}$
3. $\frac{7}{4}$
5. $\dfrac{3a^2}{4b^3}$
7. $\frac{3}{4}m^4n^3$
9. $\dfrac{1}{a + b}$
11. $\dfrac{x - 1}{x + 1}$
13. $\dfrac{x - 7}{x - 2}$
15. $\dfrac{a - 2}{a - 3}$
17. $\dfrac{a - b}{a - 2b}$
19. $\dfrac{6xy + 2y}{6xy + 3x}$
21. $-\dfrac{m - 2}{5 + 3m}$
23. $\dfrac{mn + 1}{mn}$
25. $\dfrac{x(xy - 1)}{y}$
27. $\dfrac{a^2 + 1}{a + 3}$

Section 64, Page 119

1. $\frac{1}{6}$
3. $\dfrac{20}{3ab}$
5. $\frac{14}{5}$
7. $-\dfrac{2x - y}{x - y}$
9. $\dfrac{x^2}{4}$
11. $\dfrac{-1}{x + y}$
13. $\dfrac{18(2a - 1)}{7(3a^2 - 1)}$
15. $\dfrac{x(x - a)}{2a(a + x)}$
17. $\dfrac{1}{x^2 + y^2}$
19. $4(4a - 7)$
21. 8.
23. $\dfrac{1}{x}$
25. 1.
27. $\dfrac{12(x - 6)}{x(x + 1)(x + 6)}$

Section 65, Page 122

1. $\dfrac{15a + 10b + 6c}{30}$ 3. $\dfrac{25 - 15x^2 + 28x}{20x^3}$ 5. $\dfrac{c + b}{bc}$

7. $\dfrac{3}{2(x - 2y)}$ 9. $\dfrac{4a - 3}{2(a - 1)(2a - 1)}$ 11. $\dfrac{-6}{(x - 5)(x - 2)}$

13. $\dfrac{3x^2 - 12x + 11}{(x - 1)(x - 2)(x - 3)}$ 15. $\dfrac{1 - 23m}{2(m - 2)(2m + 1)}$ 17. $\dfrac{2x^2 - 8x + 6}{1 - 4x^2}$

19. $\dfrac{2m^3 - m^2 + 2m - 5}{(m^2 - 1)(m^2 + 1)}$ 21. $\dfrac{-2}{(x + 1)(x + 2)}$

23. $\dfrac{x}{(x - y)(x + y)(x + 2y)}$ 25. $\dfrac{2x^2 + 2x - 3}{(x - 1)(x + 1)}$

27. $\dfrac{-2(2x + 1)}{(x - 1)(x - 2)(x - 3)}$ 29. $\dfrac{2}{(a + 1)(a - 1)}$

Section 66, Page 124

1. $\dfrac{a}{bc}$ 3. $\dfrac{y^2}{6}$ 5. $\dfrac{4m + 3n}{3m + 4n}$ 7. $\dfrac{a + 1}{a}$

9. x 11. $\dfrac{5}{6y}$ 13. $\dfrac{x - 2}{x + 1}$ 15. $\dfrac{m + n}{m - n}$

17. $-\frac{2}{3}y$ 19. $\dfrac{1}{m + n}$ 21. $m + n - 2x$

Section 67, Page 127

1. $x = 3$ 3. $x = 9\frac{5}{6}$ 5. $c = 6$ 7. $x = -\frac{9}{7}$
9. $x = -\frac{3}{7}$ 11. $(2, -1)$ 13. $(15, 22)$ 15. $(3, -3)$
17. $(3, -3)$ 19. $(-1, \frac{1}{9})$ 21. $(0, \frac{1}{3})$ 23. $(\frac{5}{19}, \frac{5}{17})$
25. $(\frac{5}{6}, 1)$

Section 68, Page 131

1. $78\frac{3}{4}$ min. 3. 20 hr., 30 hr. 5. $17\frac{1}{7}$ min., 24 min.
7. 4 min. 9. 252 hr. 11. 30 days
13. $21\frac{1}{19}$

Section 70, Page 135

1. $4\sqrt{2}$ 3. $5\sqrt{3}$ 5. $4\sqrt{3}$ 7. 9
9. $\sqrt{39}$ 11. $4x^2$ 13. 6 15. $\frac{1}{6}\sqrt{30}$
17. $\frac{1}{21}\sqrt{105}$ 19. $\frac{1}{6}\sqrt{78}$ 21. $\frac{3}{4}\sqrt{10}$ 23. $\frac{1}{9}\sqrt{6}$
25. $\dfrac{2m^2n}{15a^2}\sqrt{6a}$

Section 71, Page 135

1. $2\sqrt{2}$ 3. $5\sqrt{5}$ 5. $8\sqrt{5}$ 7. $21\sqrt{7}$
9. $\frac{1}{6}\sqrt{6} + \frac{1}{2}\sqrt{2}$ 11. $\frac{8}{3}\sqrt{3} - 2\sqrt{6}$ 13. $\dfrac{x + y - 1}{xy}\sqrt{xy}$
15. $(n + 2ma)\sqrt{n}$

Section 72, Page 136

1. $6 + 2\sqrt{3}$ **3.** $6 - 6\sqrt{2} + 2\sqrt{6}$ **5.** $10(\sqrt{5} - 1 - \sqrt{2})$
7. $54\sqrt{2} + 24 + 20\sqrt{3}$ **9.** $7 + 13\sqrt{15}$ **11.** $30 + 12\sqrt{6}$
13. $a - b$ **15.** $2xy + (x + y)\sqrt{xy}$ **17.** $7 - 5\sqrt{2}$
19. $5 + 3\sqrt{3}$ **21.** 16 **23.** 13

Section 73, Page 138

1. $9\sqrt{7} - 2\sqrt{5} + \sqrt{3}$ **3.** $9 + \sqrt{15} - 2\sqrt{6}$
5. $\sqrt{15} - 3\sqrt{65} - 4\sqrt{35}$ **7.** $\frac{5}{4} + \frac{1}{4}\sqrt{21} - \frac{1}{4}\sqrt{3}$
9. $2(\sqrt{3} - \sqrt{2}$ **11.** $3\sqrt{3} - 4\sqrt{2}$
13. $5 + 2\sqrt{6}$ **15.** $\dfrac{\sqrt{10} + 5}{-3}$
17. $\dfrac{\sqrt{m} + \sqrt{n}}{m - n}$ **19.** $-\frac{1}{50}\sqrt{10}$
21. $\dfrac{15 + 6\sqrt{3}}{13}$ **23.** $-3\sqrt{2} - \sqrt{3}$

Section 74, Page 140

1. 2.236 **3.** 7.886 **5.** 269.9 **7.** 18.29
9. .7924 **11.** 3514 **13.** .1327 **15.** 3.569
17. 2.725 **19.** 1.414

Section 75, Page 142

1. 8.775 **3.** 10.95 **5.** 12.17 **7.** 10.82
9. 17.24 **11.** 81.24 **13.** .9592 **15.** 1058
17. 1.658 **19.** 3.550 **21.** 1.918 **23.** .102
25. 4.038

Section 76, Page 144

Answers listed are those which will be obtained by interpolation in the 4-place table.

1. 5.338 **3.** 5.719 **5.** 9.285 **7.** 7.809 **9.** 3.354
11. 6.421 **13.** 3.270 **15.** 5.604

Section 77, Page 147

1. 2.049 **3.** 13.04 **5.** 15.52 **7.** 45.83
9. .1924 **11.** 1.110 **13.** 2.674 **15.** 210.5
17. 56.30 ft. **19.** 72.21 ft. **21.** 27.58 ft.
23. 4.243, 19.24, 23.20, and 29.83 in.

Section 78, Page 149

1. $x = 0, 9$ **3.** $x = \pm\sqrt{6}$ **5.** $x = 0, \frac{2}{3}$ **7.** $x = \pm 4$
9. $x = \pm\frac{5}{4}$ **11.** $x = \pm 2$ **13.** $x = \pm 1$ **15.** $x = \pm a\sqrt{2}$
17. $x = \pm 2a\sqrt{5}$

Section 79, Page 151

1. 9 **3.** 4 **5.** $\frac{9}{4}$ **7.** a^2 **9.** $\frac{9}{64}a^2$
11. $x = 2, 6$ **13.** $x = 1, -4$ **15.** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{41}$
17. $x = 5 \pm \frac{1}{2}\sqrt{182}$ **19.** $x = \frac{3}{4}, -4\frac{1}{2}$ **21.** $x = a(-3 \pm \sqrt{19})$
23. $x = 3a, -2a$ **25.** $x = \frac{5}{4}a, -3a$

Section 80, Page 153

1. $x = 3, 4$ **3.** $x = 6, 1$ **5.** $x = 4, 2$ **7.** $x = 2\frac{1}{2}, -4$
9. $x = \frac{3}{2}, -\frac{5}{3}$ **11.** $x = 3a, -2a$ **13.** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{41}$
15. $x = \frac{3}{2} \pm \frac{3}{2}\sqrt{3}$ **17.** $x = -\frac{3}{2} \pm \frac{1}{6}\sqrt{38}$
19. $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ **21.** $x = \pm\sqrt{17}$

Section 84, Page 161

Answers are accurate to the nearest tenth.

1. $(6, 4), (-\frac{3}{2}, \frac{1}{4})$ **3.** $(3, 1), (-.5, -6)$ **5.** $(2.8, 2.4), (-3, -.5)$
7. $(5, 5), (0, 0)$ **9.** $(3.3, 1.5)(-.8, -6.5)$
11. $(2.1, 4.1)(-3.4, -1.4)$

Section 85, Page 162

Answers are accurate to nearest tenth.

1. $(3, 1)$ **3.** $(4.5, 2.2), (-4.5, 2.2)$ **5.** $(4.2, 2.8), (-.6, 5)$
7. $(3.6, 3.5), (3.6, -3.5), (-3.6, 3.5), (-3.6, -3.5)$
9. $(2.9, -1.1), (-.8, 5.8)$
11. $(2.7, 2.5), (2.7, -2.5), (-2.7, 2.5), (-2.7, -2.5)$
13. $(2.9, 1.7)$ **15.** $(3.3, 2.2), (-3.3, 2.2)$
17. $(3.7, 1.4), (-2.1, 3.4), (-.4, 4), (2.8, -2.9)$
19. $(3, 2.6), (3, -2.6), (-3, 2.6), (-3, -2.6)$
21. $(-2.2, 4), (0, -5), (2.6, -3.3), (3.5, .4)$
23. $(3, 2.6), (3, -2.6), (-3, 2.6), (-3, -2.6)$

Section 86, Page 166

1. $(\sqrt{5}, 5 - \sqrt{5}), (-\sqrt{5}, 5 + \sqrt{5})$ **3.** $(5, 2), (-4, -\frac{5}{2})$
5. $(4, -3), (\frac{8}{3}, -\frac{1}{3})$ **7.** $(\frac{1}{2}, \frac{1}{2})$
9. $(4, 8)(-4, -8)$ **11.** $(2a, a), (-a, -2a)$

Answers to 13–24, inclusive, should be checked by referring to graphical solutions obtained in Section 84.

Section 87, Page 168

1. $(3, 2), (3, -2), (-3, 2)\ (-3, -2)$
3. $(3, 2), (3, -2), (-3, 2), (-3, -2)$
5. $(6, 5), (6, -5), (-6, 5), (-6, -5)$
7. $(3, 3), (3, -3), (-3, 3), (-3, -3)$
9. $(\sqrt{2}, \sqrt{13}), (\sqrt{2}, -\sqrt{13}), (-\sqrt{2}, \sqrt{13}), (-\sqrt{2}, -\sqrt{13})$
11. $(\frac{1}{5}\sqrt{195}, \frac{2}{5}\sqrt{15}), (\frac{1}{5}\sqrt{195}, -\frac{2}{5}\sqrt{15}), (-\frac{1}{5}\sqrt{195}, \frac{2}{5}\sqrt{15})(-\frac{1}{5}\sqrt{195},$
$-\frac{2}{5}\sqrt{15})$
13. $(2, 4), (2, -4), (-2, 4), (-2, -4)$
15. $(\sqrt{2} + \sqrt{19}, -2\sqrt{2}), (\sqrt{2} - \sqrt{19}, -2\sqrt{2}), (-\sqrt{2} + \sqrt{19}, 2\sqrt{2}),$
$(-\sqrt{2} - \sqrt{19}, 2\sqrt{2})$

Section 88, Page 171

1. 24 **3.** 10 by $5\frac{1}{2}$ in. **5.** 8 and 9 **7.** 7 and 10
9. 12 suits **11.** 12 and 4 **13.** 9 and 7, or -1 and 17
15. 5 and $\sqrt{7}$ **17.** 160 m.p.h. **19.** 76 m.p.h., 1140 miles
21. 16 and 10 in., or 8 and 2 in.

Section 89, Page 174

1. $x = 3, -3, 2, -2$ **3.** $x = 1, -1, \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}$
5. $x = \pm\frac{1}{2}\sqrt{10 \pm 2\sqrt{5}}$ **7.** $\pm\frac{1}{3}\sqrt{12 \pm 3\sqrt{7}}$
9. $\pm m\sqrt{3}$

Section 90, Page 177

1. $(1, 1), (-1, -1)$ **3.** $(2\sqrt{2}, \sqrt{2}), (-2\sqrt{2}, -\sqrt{2})$
5. $(\frac{15}{14}\sqrt{14}, -\frac{1}{14}\sqrt{14}), (-\frac{15}{14}\sqrt{14}, \frac{1}{14}\sqrt{14})$
7. $(8, 2\sqrt{5}), (8, -2\sqrt{5}), (-4, 2\sqrt{2}), (-4, -2\sqrt{2})$
9. $(-1 + 2\sqrt{3}, \sqrt{8 - 4\sqrt{3}}), (-1 + 2\sqrt{3}, -\sqrt{8 - 4\sqrt{3}}),$
$(-1 - 2\sqrt{3}, \sqrt{8 + 4\sqrt{3}}), (-1 - 2\sqrt{3}, -\sqrt{8 + 4\sqrt{3}})$

Section 91, Page 181

A

1. $(3, 1), (6, 4), (0, 0)$ **3.** $(3, 1), (-3, -1), (\frac{3}{2}, 2), (-\frac{3}{2}, -2)$
5. $(\frac{6}{29}\sqrt{29}, \frac{30}{29}\sqrt{29}), (-\frac{6}{29}\sqrt{29}, -\frac{30}{29}\sqrt{29}), (\frac{3}{2}\sqrt{2}, -3\sqrt{2}), (-\frac{3}{2}\sqrt{2}, 3\sqrt{2})$

or

$(1.11, 5.57), (-1.11, -5.57), (2.12, -4.24), (-2.12, 4.24)$

7. $\left[\pm\frac{6(2 + \sqrt{3})}{\sqrt{29 + 16\sqrt{3}}}, \pm\frac{6}{\sqrt{29 + 16\sqrt{3}}}\right],$
$\left[\pm\frac{6(2 - \sqrt{3})}{\sqrt{29 - 16\sqrt{3}}}, \pm\frac{6}{\sqrt{29 - 16\sqrt{3}}}\right]$

or

$(2.97, .80), (-2.97, -.80), (1.41, 5.29), (-1.41, -5.29)$

9. $\left[\pm(\frac{5}{2} + \frac{5}{6}\sqrt{21})\sqrt{\frac{6}{11 + \sqrt{21}}}, \pm 5\sqrt{\frac{6}{11 + \sqrt{21}}}\right],$
$\left[\pm(\frac{5}{2} - \frac{5}{6}\sqrt{21})\sqrt{\frac{6}{11 - \sqrt{21}}}, \pm 5\sqrt{\frac{6}{11 + \sqrt{21}}}\right]$

or

$(3.92, 3.10), (-3.92, -3.10), (1.28, -4.83), (1.28, 4.83)$

B

11. $(0, 0), (-5, 5), (2, \frac{4}{5})$ **13.** $(5, 1), (-5, -1), (\frac{5}{2}, 2), (-\frac{5}{2}, -2)$
15. $\left(\frac{5}{\sqrt{13}}, -\frac{5}{\sqrt{13}}\right), \left(\frac{-5}{\sqrt{13}}, \frac{5}{\sqrt{13}}\right), \left(\frac{5}{\sqrt{27}}, \frac{25}{\sqrt{27}}\right), \left(\frac{-5}{\sqrt{27}}, \frac{-25}{\sqrt{27}}\right)$

17. $\left[\pm\sqrt{5(2+\sqrt{3})},\ \pm\sqrt{\dfrac{5}{2+\sqrt{3}}},\ \left[\pm\sqrt{5(2-\sqrt{3})},\ \pm\sqrt{\dfrac{5}{2-\sqrt{3}}}\right]\right.$

19. $\left[\pm\dfrac{4(2+\sqrt{5})}{\sqrt{10+4\sqrt{5}}},\ \pm\dfrac{4}{\sqrt{10+4\sqrt{5}}}\right],\ \left[\pm\dfrac{4(2-\sqrt{5})}{\sqrt{10-4\sqrt{5}}},\ \pm\dfrac{4}{\sqrt{10-4\sqrt{5}}}\right]$

Section 92, Page 184

1. $(4, 3), (-4, -3), (3, -4), (-3, 4)$ **3.** $(2, -1), (-2, 1)$

5. $(\frac{5}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}), (-\frac{5}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}), (3, 2), (-3, -2)$
7. $(\frac{5}{3}\sqrt{3}, \frac{5}{6}\sqrt{3}), (-\frac{5}{3}\sqrt{3}, -\frac{5}{6}\sqrt{3})$

Section 93, Page 185

1. $(\sqrt{33}, \sqrt{6}), (\sqrt{33}, -\sqrt{6}), (-\sqrt{33}, \sqrt{6}), (-\sqrt{33}, -\sqrt{6})$
3. $(\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}), (\frac{9}{17}\sqrt{17}, \frac{6}{17}\sqrt{17}), (-\frac{9}{17}\sqrt{17}, -\frac{6}{17}\sqrt{17})$
5. $(\frac{5}{2}, \frac{5}{2}), (-\frac{5}{2}, -\frac{5}{2}), (10, -5), (-10, 5)$
7. $(\sqrt{11}, \frac{1}{2}\sqrt{26}), (\sqrt{11}, -\frac{1}{2}\sqrt{26}), (-\sqrt{11}, \frac{1}{2}\sqrt{26}), (-\sqrt{11}, -\frac{1}{2}\sqrt{26})$
9. $(\frac{1}{2}\sqrt{10}, -\frac{1}{2}\sqrt{10}), (-\frac{1}{2}\sqrt{10}, \frac{1}{2}\sqrt{10}), (2, -1), (-2, 1)$
11. $(6, 2), (-6, -2), (2, 6), (-2, -6)$
13. $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}), (\frac{1}{2}, -3), (-\frac{1}{2}, 3)$

15. $(2\sqrt{6}, 1), (-2\sqrt{6}, 1), (0, -5)$ **17.** $(2, -1), (13, -5)$

19. $(1, 2), (-1, -2), (\sqrt{2}, -\frac{1}{2}\sqrt{2}), (-\sqrt{2}, \frac{1}{2}\sqrt{2})$

Section 94, Page 189

1. $7i$ **3.** $-3i$ **5.** $2i\sqrt{3}$ **7.** $2ai$ **9.** $-2a^2i\sqrt{5}$
11. $2x^2i\sqrt{6x}$ **13.** $x = \pm 6i$ **15.** $x = \pm 3i\sqrt{5}$ **17.** $x = \pm 2i\sqrt{2}$
19. i **21.** i **23.** $-i$ **25.** $-i$

Section 95, Page 191

1. $x = 2$, $y = 3$ **3.** $x = -2$, $y = -2$ **5.** $x = 8$, $y = 3$ **7.** $x = 2, 3$, $y = 1, 2$
9. $(2, 3), (2, -3), (-2, 3), (-2, -3)$

Section 96, Page 193

1. $5 - 12i$ **3.** $-13i$ **5.** $\frac{20}{29} - \frac{21}{29}i$

7. $\frac{14}{13} + \frac{5}{13}i$ **9.** $.14 - .52i$ **11.** $x = 3 + i$, $x = 3 - i$ **13.** $x = 1 + i\sqrt{2}$, $x = 1 - i\sqrt{2}$

15. $x = 3$ **17.** $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, $x = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ **19.** $1 + \frac{1}{3}i\sqrt{3}$, $1 - \frac{1}{3}i\sqrt{3}$

Section 97, Page 195

1. Real, unequal **3.** Imaginary **5.** Real, unequal

7. If $m < 2$, the roots are real and unequal
If $m = 2$, the roots are real and equal
If $m > 2$, the roots are complex

9. If a is between $\frac{3}{4}$ and $-\frac{3}{4}$, the roots are real and unequal
If a equals $\frac{3}{4}$ or $-\frac{3}{4}$, the roots are real and equal
If a is not between $\frac{3}{4}$ and $-\frac{3}{4}$ or equal to one of them, the roots are complex

11. Real, unequal

13. The roots are real if m equals either $\frac{25}{4}$ or $-\frac{25}{4}$, or if its value is between them

15. If $mn \geqq -\frac{49}{4}$, the roots are real

Section 98, Page 199

1. See B, Fig. 28, p. 162

3. $(2\sqrt{5}, 4)$, $(-2\sqrt{5}, 4)$, $(i\sqrt{28}, -8)$, $(-i\sqrt{28}, -8)$. Two real solutions; see A, Fig. 28

5. $(5i, -5i)$, $(-5i, 5i)$

7. $(\frac{4}{3} + \frac{2}{3}i\sqrt{2}, -\frac{2}{3} - \frac{4}{3}i\sqrt{2})$, $(\frac{4}{3} - \frac{2}{3}i\sqrt{2}, -\frac{2}{3} + \frac{4}{3}i\sqrt{2})$
See F, Fig. 28, for graph of $x^2 - y^2 = 4$

9. $(5, -3)$, $(-1, -9)$

11. $(-4, 0)$, $(\frac{14}{3}, \frac{2}{3}i\sqrt{13})$, $(\frac{14}{3}, -\frac{2}{3}i\sqrt{13})$. One double real solution.

13. $(3, 3)$, $(-3, -3)$, $(i\sqrt{3}, -i\sqrt{3})$, $(-i\sqrt{3}, i\sqrt{3})$

15. $(-\frac{3}{2} + \frac{1}{2}i, \frac{3}{2} + \frac{1}{2}i)$, $(-\frac{3}{2} - \frac{1}{2}i, \frac{3}{2} - \frac{1}{2}i)$. No real solutions.

17. $(4, \frac{1}{2})$, $(-4, -\frac{1}{2})$, $(i, -2i)$, $(-i, 2i)$

19. $(\frac{1}{5}\sqrt{85}, \frac{6}{5})$, $(-\frac{1}{5}\sqrt{85}, \frac{6}{5})$, $(i\sqrt{3}, -2)$, $(-i\sqrt{3}, -2)$

Section 100, Page 203

1. a^8	3. x^6y^8z	5. x^6	7. $-\frac{1}{8}x^3$	9. $9x^4y^6$
11. $\frac{y^4}{x^8}$	13. x^5y^{13}	15. $\frac{1}{8}\frac{a^3}{b^3}$	17. $6\frac{z^7}{y^6}$	19. $\frac{64}{81}$

Section 101, Page 205

1. 1	3. $\frac{1}{9}$	5. 27	7. $\frac{1}{2}$	9. $441\frac{4}{9}$
11. x^2	13. a^5	15. m^5	17. $\frac{z^3}{xy^2}$	19. $\frac{2b^2}{3a^5}$
21. $\frac{x+y}{x-y}$	23. $\frac{5m}{n}$	25. $\frac{1}{a^2} - \frac{1}{b^2}$	27. $\frac{-(m+n)}{mn(m-n)}$	29. x^{m+3}

31. $\frac{x^{2b}}{x^{2a}}$ or x^{2b-2a} 33. $\frac{x^m}{y^m}$ or $\left(\frac{x}{y}\right)^m$

Section 102, Page 206

1. $\frac{1}{2}$	3. a^2	5. $\frac{m^2}{n^2}$	7. $2\sqrt[3]{9}$	9. .2
11. $20\sqrt[3]{4}$	13. $.5x\sqrt[3]{x^2}$			

Section 103, Page 208

1. $\sqrt[3]{a}$, $\sqrt{a}$, $\sqrt[3]{a^2}$, $\sqrt[10]{a^7}$, $\sqrt[10]{a^9}$, $\frac{1}{x}\sqrt{x}$, $\frac{1}{x}\sqrt[3]{x}$

3. $2\sqrt{x}$, $\sqrt{2x}$, $8\sqrt[3]{x}$, $2\sqrt[3]{x}$

5. $x^{\frac{1}{5}}$, $x^{\frac{4}{3}}$, $x^{\frac{9}{2}}$, $(x+y)^{\frac{1}{2}}$, $(x+y)^{\frac{2}{3}}$ 7. 6 9. 8 11. 32

13. 4	15. $\frac{1}{8}$	17. 2	19. .2	21. .25	23. $x^{\frac{5}{6}}$
25. a	27. $m^{\frac{13}{6}}$	29. $a^{\frac{1}{3}}$	31. $m^{\frac{1}{6}}$	33. $x^{\frac{1}{2}}$	35. $x^{4.33}$

37. $10^{1.598}$ 39. $10^{0.90}$ 41. x^4 43. m^2n^3 45. $x^{-\frac{1}{12}}$

47. $x^{\frac{3}{4}}$ 49. $a^{\frac{3}{4}}b^{-\frac{1}{4}}$ 51. $x^{\frac{7}{12}}y^{\frac{3}{4}}$ 53. $10^{0.66}$

Section 104, Page 210

1. $\sqrt[6]{108}$	**3.** $\sqrt[12]{6912}$	**5.** $\sqrt[6]{12}$	**7.** $a\sqrt[21]{a^8}$
9. $y\sqrt[12]{200x^9y^7}$	**11.** $xy\sqrt[10]{45}$		**13.** $\sqrt[6]{\frac{27}{4}}$ or $\frac{1}{2}\sqrt[6]{432}$
15. $\sqrt[12]{\frac{256}{27}}$	**17.** $\dfrac{1}{\sqrt[21]{a}}$	**19.** $\sqrt[15]{x}$	**21.** $\dfrac{1}{\sqrt[6]{a^5b^5}}$
23. $\sqrt[10]{\frac{25}{3}x^2y^4}$			

Section 107, Page 215

1. 2.3×10^3	**3.** 8.439×10^3	**5.** 8.72×10^0
7. 6.34×10^{-2}	**9.** 2.16×10^{-1}	**11.** 6.234×10^5
13. 6.246×10^7	**15.** 6.142×10^{-4}	**17.** 2.4323×10^4
19. 3.124×10^1		

Section 108, Page 216

1. $10^{1.4099}$	**3.** $10^{.5821-3}$	**5.** $10^{.1072-1}$
7. $10^{.0453-3}$	**9.** $10^{2.3010}$	**11.** $10^{7.8028}$
13. $10^{.7910-4}$	**15.** $10^{.2380-2}$	**17.** $10^{.8633-5}$
19. $10^{0.3729}$		

Section 109, Page 218

1. $10^{1.6868}$	**3.** $10^{1.8044}$	**5.** $10^{.7954-3}$
7. $10^{2.0944}$	**9.** $10^{4.0260}$	**11.** $10^{.7892-1}$
13. $10^{1.2701}$	**15.** $10^{2.3288}$	**17.** $10^{3.0139}$
19. $10^{1.9946}$		

Section 110, Page 219

1. 902.0	**3.** .3650	**5.** 40,600
7. .06370	**9.** 83.44	**11.** .5634
13. 372.3	**15.** .001963	**17.** .05188
19. .0003532		

Section 111, Page 221

1. 817.2	**3.** 3,287	**5.** 766.3	**7.** 3.065
9. .07602	**11.** .0002488	**13.** .002585	**15.** 15.34
17. 187.2	**19.** 8.868	**21.** .07553	**23.** 11,290
25. 756.5	**27.** 6.452	**29.** 531.1	**31.** 625.7
33. 31.20	**35.** 156.9		

Section 112, Page 224

1. 2.3464	**3.** .8733 − 3	**5.** 0.9124	**7.** 7.8030
9. .6647 − 1	**11.** 0.7580	**13.** 1.8147	**15.** .2365 − 2
17. 2,641	**19.** .03702	**21.** .001836	**23.** .3380
25. 7.058	**27.** 101.8	**29.** 547.2	

Section 113, Page 228

1. 72.65	**3.** 33.62	**5.** 87.70	**7.** 1,665
9. 1,139	**11.** 58.42	**13.** .000003889	**15.** 2.210
17. .0005239	**19.** 33.52	**21.** 1,113	**23.** 5.555
25. 1,537	**27.** 203.3	**29.** 6.677	**31.** 8.436
33. 1,886	**35.** 6.948	**37.** 533.4	**39.** 625.7

Section 115, Page 229

1. $x = 2$ **3.** $x = 3$ **5.** $y = 9$ **7.** $y = 81$
9. $x = 7$ **11.** $a = 7$ **13.** $y = 10$ **15.** $a = 100$
17. $\log_a c = b$ **19.** $\log_5 625 = 4$ **21.** $\log_4 (y - 7) = x + 1$
23. $\log_{r-s} z = t$

Section 116, Page 231

1. 3.482 **3.** 4.782 **5.** 2.837 **7.** 1.965
9. 1.174 **11.** 3.239 **13.** 3.203 **15.** −2.33
17. 1.244 **19.** 8.656 **21.** 9.696 **23.** −2.224

Section 117, Page 234

1. $\frac{6}{7}$ **3.** $\frac{7}{4}$ **5.** $\frac{5}{2}$ **7.** $\frac{8}{3}$ **9.** 7:12:18
11. $\frac{625}{762}$ **13.** $\frac{11}{30}$ **15.** $\frac{9}{16}$
17. 20% corn, $33\frac{1}{3}$% barley, by volume
19. 21:30:28 **21.** 60, 80, and 100 **23.** 14.66 gal.
25. .1083

Section 118, Page 235

5. 15 **9.** $\pm\sqrt{6}$ **11.** 7 ft. **13.** $8\frac{8}{15}$ ft., $7\frac{7}{15}$ ft.
15. 7.795 sq. ft. **17.** $755\frac{5}{8}$ lb. **19.** 1,150

Section 119, Page 240

1. $a = 36$ **3.** 733.2 **5.** 87 **7.** .8818 lb.
9. 71.4 ohms **11.** 36 ft. **13.** 80 ft. per sec.
15. 50.23 cu. ft. **17.** 40.20 cu. in.

Section 121, Page 245

Ex.	a	b	c	A	B
1	30.32 ft.	39.76 ft.	—	—	52°40′
3	228.7 ft.	—	303.8 ft.	48°50′	—
5	23.84 ft.	7.52 ft.	—	72°30′	—
7	—	32.21 ft.	37.92 ft.	—	58°10′
9	—	—	—	63°30′*	26°30′*
11	—	—	—	31°10′*	58°50′*
13	—	—	—	86°50′*	3°10′*
15	40.20 ft.	25.12 ft.	—	58°	—
17	1,763 ft.	—	5,919 ft.	—	72°40′

* Nearest value listed in the table.

Section 122, Page 247

1. .9787 **3.** .9068 **5.** .8942 **7.** 1.924
9. .5908 **11.** .9689 **13.** 1.942 **15.** .1866
17. 1.083 **19.** 1.125 **21.** .3727 **23.** .6196

Section 123, Page 248

1. 8°13′ **3.** 21°53′ **5.** 54°31′ **7.** 34°17′
9. 77°25′ **11.** 23°36′ **13.** 13°35′ **15.** 25°58′
17. 61°8′ **19.** 55°18′ **21.** 39°10′ **23.** 45°22′

Section 124, Page 250

1. .9678 − 1 **3.** 0.5580 **5.** .8749 − 1 **7.** .3449 − 1
9. .9150 − 2 **11.** 14°22′ **13.** 49°23′ **15.** 33°3′
17. 74°37′ **19.** 4°22′

Ex.	a	b	c	A	B
21	350.3	455.9	—	—	52°26′
23	38.95 ft.	—	51.83 ft.	48°43′	—
25	24.50 ft.	7.748 ft.	—	72°27′	—
27	—	88.76 ft.	93.05 ft.	—	72°33′
29	222.5 ft.	—	—	78°39′	11°21′
31	—	—	176.8 ft.	7°52′	82°8′

33. 25°22′ **35.** 3.237 minutes **37.** 68°50′ **39.** 11°19′

41. 273.4 m.p.h. **43.** 113.4 miles **45.** 15.06 minutes

Section 127, Page 259

1. $x = 436$ **3.** $x = 135\frac{2}{3}$ **5.** $x = \pm 2$
7. $x = 7$ **9.** $x = 9$ **11.** $x = 2$
13. No roots **15.** $x = 5$

Section 130, Page 262

1. Arithmetic, 19 **3.** Geometric, 1,250 **5.** Neither
7. Arithmetic, 7 **9.** Geometric, $9\sqrt{3}$ **11.** 1
13. 7 **15.** $\frac{1}{2}$ **17.** 4
19. 1, −8 **21.** 24 **23.** −6
25. 11 **27.** 0 **29.** $\frac{1}{2} \pm \frac{1}{6}i\sqrt{39}$

Section 131, Page 264

1. 40 **3.** −9 **5.** 41 **7.** $40\frac{1}{2}$ **9.** −8.1

Section 132, Page 265

1. 19, $22\frac{1}{5}$, $25\frac{2}{5}$, $28\frac{3}{5}$, $31\frac{4}{5}$, 35 **3.** 6, $6\frac{1}{3}$, $6\frac{2}{3}$, 7
5. 8.4, 8.6, 8.8, 9.0, 9.2, 9.4, 9.6, 9.8, 10.0, 10.2, 10.4, 10.6, 10.8
7. $27\frac{1}{2}$ **9.** -25, -8, 9, 26, 43

Section 133, Page 267

1. 465 **3.** 126 **5.** $123\frac{1}{2}$ **7.** 185.4 **9.** 16.1
11. 20 **13.** $a = 8$, $d = 4$ **15.** $n = 11$, $S = 269\frac{1}{2}$
17. $l = 5\frac{1}{2}$, $S = 2$ **19.** $l = 11.8$, $d = -1.3$

Section 134, Page 268

1. 64 **3.** 64 **5.** 10^6 **7.** 4.151 **9.** ± 3
11. 32, .8

Section 135, Page 269

1. 6 **3.** ± 7.483 **5.** 13.10, 34.34 **7.** ± 6.593, 14.49, ± 31.86

Section 136, Page 271

1. $511\frac{1}{2}$ **3.** $\frac{1533}{64}$, or 23.953 **5.** $30 + 15\sqrt{2}$, or 51.21
7. 4.096 **9.** $a = 1$, $n = 9$ **11.** $l = \frac{1}{24}$, $n = 6$

Section 137, Page 273

1. 2 **3.** $12\frac{1}{2}$ **5.** $\frac{3}{2}x$ **7.** $-3\frac{3}{5}$ **9.** $2\frac{2}{5}$ **11.** $\frac{2}{3}$
13. $\frac{1}{9}$ **15.** $\frac{47}{90}$ **17.** $1\frac{357}{1100}$ **19.** $2\frac{4}{11}$ **21.** $\frac{47}{333}$

Section 138, Page 274

1. \$15.50 **3.** \$2,412 **5.** 100 ft. **7.** 90 ft.
9. 2,304 ft. **11.** 353.44 ft. **13.** 168 posts **15.** \$1,697
17. 11.90 years, 18.93 years **19.** .4786 **21.** 1.73

Section 139, Page 278

1. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
3. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
5. $c^8 + 8c^7d + 28c^6d^2 + 56c^5d^3 + 70c^4d^4 + 56c^3d^5 + 28c^2d^6 + 8cd^7 + d^8$
7. $a^{15} + 15a^{14}b + 105a^{13}b^2 + 455a^{12}b^3 + \cdots$
9. $x^{17} + 17x^{16}y + 136x^{15}y^2 + 680x^{14}y^3 + \cdots$
11. $c^{20} + 20c^{19}d + 190c^{18}d^2 + 1140c^{17}d^3 + \cdots$

Section 140, Page 280

1. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$
3. $a^6 - 12a^5 + 60a^4 - 160a^3 + 240a^2 - 192a + 64$
5. $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$
7. $16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4$
9. $1 - 10y + 40y^2 - 80y^3 + 80y^4 - 32y^5$
11. 1.006012008 **13.** .991026973
15. $x^6 + 3x^4y + 3x^2y^2 + y^3$
17. $a^{35} + 7a^{30}b + 21a^{25}b^2 + 35a^{20}b^3 + 35a^{15}b^4 + 21a^{10}b^5 + 7a^5b^6 + b^7$
19. $\frac{x^4}{y^4} + 4\frac{x^2}{y^2} + 6 + 4\frac{y^2}{x^2} + \frac{y^4}{x^4}$ **21.** 1.0181 **23.** .9081

Section 141, Page 281

1. $\frac{1}{x} - \frac{y}{x^2} + \frac{y^2}{x^{\circ}} - \frac{y^3}{x^4} + \cdots$ **3.** $1 - 5x + 15x^2 - 35x^3 + \cdots$

5. $a^{-\frac{1}{2}} - \frac{1}{2}a^{-\frac{3}{2}}b + \frac{3}{8}a^{-\frac{5}{2}}b^2 - \frac{5}{16}a^{-\frac{7}{2}}b^3 + \cdots$

7. $a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}b - \frac{1}{9}a^{-\frac{4}{3}}b^2 + \frac{4}{81}a^{-\frac{7}{3}}b^3 + \cdots$

9. $x^{-7} + 7x^{-8}y + 28x^{-9}y^2 + 84x^{-10}y^3 + \cdots$ **11.** $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

13. .998003992 **15.** 1.003992032 **17.** 1.00029982 **19.** .996048

Section 142, Page 282

1. $126x^5y^4$ **3.** $792x^5y^7$ **5.** $14{,}784x^6$ **7.** $3003x^5$ **9.** .00002016

Section 143, Page 285

1. 1.0004 **3.** 2.0008 **5.** .33184 **7.** 2.0010 **9.** 1.9996
11. .33347 **13.** .00003 **15.** .000045

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